

An Adjunction Between the Lens and the Int Construction in Categorical Game Semantics

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Abstract

Bidirectional information flow is a central concept in both categorical game semantics and practical functional programming. In game semantics, the Joyal-Street-Verity *Int* construction provides a canonical way to build compact closed categories from traced monoidal categories, elegantly modeling bidirectional interaction and feedback loops. Conversely, in functional programming, the Lens construction offers a robust, compositional framework for bidirectional state access and update.

This work-in-progress explores the structural relationship between these two paradigms. We propose an adjunction between the categories arising from the Lens and *Int* constructions. By formally connecting the state-based approach of the Lens construction with the traced monoidal approach of the *Int* construction, we aim to bridge abstract game-semantic models with compositional theories of bidirectional flow.

In this talk, we will outline the core functorial mappings between these categories, present the ongoing proofs of the adjunction, and discuss its potential applications to the foundations of game semantics.

1 Introduction

The study of bidirectional data flow spans categorical logic, game semantics, and functional programming. In game semantics, the *Int* construction [?] provides a canonical geometric setting for bidirectional interaction via feedback loops. Conversely, Lenses are standard abstractions for bidirectional state access. However, the string diagrammatic representation of Lenses often lacks the intuitive geometric properties found in the *Int* construction. Motivated by the search for a geometrically superior alternative, we propose a formal connection between these paradigms. Specifically, we investigate a pointwise adjunction between the category of Lenses and the category resulting from the *Int* construction over a traced cartesian closed category.

2 Background: The Lens and Int Constructions

Throughout this abstract, all monoidal categories are assumed to be strict. Let $\mathcal{C}$ be a Cartesian monoidal category.

Definition 1 (Lens). *The category of Lenses, denoted as $\mathbf{Lens}(\mathcal{C})$, is a symmetric monoidal category whose objects are pairs $\langle A^+, A^- \rangle$ of objects in $\mathcal{C}$. A morphism from $\langle A^+, A^- \rangle$ to $\langle B^+, B^- \rangle$ is given by a pair of morphisms $\langle f^+, f^- \rangle$ where:*

$$f^+ : A^+ \rightarrow B^+, \quad f^- : A^+ \times B^- \rightarrow A^- \quad (1)$$

If $\mathcal{C}$ is a traced symmetric monoidal category, We can apply the *Int* construction to obtain a compact closed category.

Definition 2. The category $\text{Int}(\mathcal{C})$ shares the same objects as $\mathbf{Lens}(\mathcal{C})$, namely pairs $\langle A^+, A^- \rangle$. A morphism $f: \langle A^+, A^- \rangle \rightarrow \langle B^+, B^- \rangle$ is given by a single morphism in $\mathcal{C}$:

$$f: A^+ \otimes B^- \rightarrow B^+ \otimes A^- \quad (2)$$

Composition is defined by utilizing the trace over the internal feedback wires.

3 The Core Adjunction

The main contribution of this work-in-progress is the construction of an adjunction connecting the operational perspective of Lenses with the geometric, feedback-oriented perspective of the Int construction. To formulate this, we assume $\mathcal{C}$ is a traced cartesian closed category (TrCCC).

We define a strong symmetric monoidal functor $\tau_{\mathcal{C}}: \mathbf{Lens}(\mathcal{C}) \rightarrow \text{Int}(\mathcal{C})$. On objects, it is the identity mapping:

$$\tau_{\mathcal{C}}(\langle A^+, A^- \rangle) = \langle A^+, A^- \rangle \quad (3)$$

On morphisms, a lens $\langle f^+, f^- \rangle$ is mapped to a morphism in $\text{Int}(\mathcal{C})$ by utilizing the cartesian structure (duplication) of A^+ to feed information into both the forward pass f^+ and the backward pass f^- .

Conversely, leveraging the closed structure of $\mathcal{C}$, we define a functor $\rho_{\mathcal{C}}: \text{Int}(\mathcal{C}) \rightarrow \mathbf{Lens}(\mathcal{C})$. On objects, it is defined using the exponential object:

$$\rho_{\mathcal{C}}(\langle A^+, A^- \rangle) = \langle A^{+A^-}, A^- \rangle \quad (4)$$

The morphism mapping for $\rho_{\mathcal{C}}$ is constructed via currying and evaluation maps.

Theorem 1. Let $\mathcal{C}$ be a traced cartesian closed category. There exists an adjunction $\tau_{\mathcal{C}} \dashv \rho_{\mathcal{C}}$ between $\mathbf{Lens}(\mathcal{C})$ and $\text{Int}(\mathcal{C})$.

Theorem 2. The mappings $\mathcal{C} \mapsto \mathbf{Lens}(\mathcal{C})$ and $\mathcal{C} \mapsto \text{Int}(\mathcal{C})$ extend to well-defined functors $\mathbf{Lens}, \text{Int}: \text{TrCCC} \rightarrow \mathbf{SymMonCat}$. Moreover, the families of functors $\tau_{\mathcal{C}}$ and $\rho_{\mathcal{C}}$ constitute natural transformations $\tau: \mathbf{Lens} \Rightarrow \text{Int}$ and $\rho: \text{Int} \Rightarrow \mathbf{Lens}$ between these functors. Finally, these natural transformations form a pointwise adjunction $\tau \dashv \rho$.

4 Implications and Future Work

Establishing this adjunction provides a formal dictionary between two distinct mathematical treatments of bidirectionality. Specifically, it clarifies how the internal state manipulation in Lenses (via the $\rho_{\mathcal{C}}$ mapping to the exponential $A^{A'}$) corresponds to the feedback wires traced out in the Int construction.

Our immediate future work involves detailing the strictness conditions of this adjunction and exploring its higher-categorical structures (e.g., viewing it as a 2-adjunction). Furthermore, we aim to investigate how this framework can be extended to more general Optics and its implications for the game-theoretic interpretations of bidirectional flow.

References

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