# Takahashi's Gambit: Avoiding Context Lemmas in Operational Game Semantics

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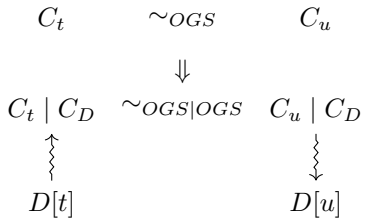
**Abstract**

Operational game semantics are presented through labelled transition systems, of which the induced bisimilarity on programs is shown to be included in CIU equivalence, a variant of contextual equivalence. Soundness of operational game semantics follows from external context lemmas which state that CIU and contextual equivalences coincide.

This talk will show how one may avoid the use of CIU equivalence entirely, by changing the initial operational configurations of terms in the operational game semantics. The construction is inspired by a proof technique for the genericity lemma due to Takahashi.

Operational game semantics (OGS) provide a flexible and robust technique to prove contextual equivalence of programs. From theoretical developments [Lai07, LS14] to automatic equivalence checkers [KLT22], these techniques have scaled up to provide semantics grounded in the theory of programming languages as well as tractable tools for effectful extensions. They are a presentation of game semantics based on labelled transition systems (LTS), borrowing some results and formalisms from process calculi [JS22].

This talk is about proving the soundness of operational game semantics. Soundness for different OGS typically follows from the same arguments. First, from two OGS bisimilar program, one shows that the composition $C_t \mid C_D$ of initial configurations $C_t$ with external contexts $C_D$ amounts to performing the operational composition $D[t]$ of these terms in the given context. The following proof sketch then shows *soundness*: for all $D$, if $t \sim_{OGS} u$ and $D[t] \Downarrow$ then $D[u] \Downarrow$.




**Soundness for CIU.** There are subtleties in this proof sketch, which partly explains why no general proof exists and the proof is remade in every new OGS LTS. Note that for the equivalence between operational termination $\Downarrow$ and termination for the composite LTS $OGS \mid OGS$, terms are required to be closed by the context $D$. What we address in this abstract is another subtlety of this equivalence: in general, it only works for *evaluation contexts* $D$, making the soundness proof a proof of soundness for CIU equivalence (the restriction of contextual equivalence to evaluation contexts). Then the OGS development typically calls for an external context lemma showing that CIU and contextual equivalence coincide. The goal of this work is to show that it is possible to avoid CIU equivalence entirely.

**Using a Context Lemma.** There exists a large literature on them but they are not so easily proven and often require different proofs for every extension of the programming language. The first context lemmas can be seen as corollaries of fully abstract presentations of contextual equivalence, *e.g.* the presentation of head contextual equivalence by Nakajima trees and Böhm's theorem led to a first proof that applicative contexts are enough to (contextually) separate terms. Easier proof techniques (not requiring a fully abstract tree characterization) then appeared, *e.g.* Howe's method [How96, Pit12].

**This Talk: How to Avoid Context Lemmas.** In the talk, we will explain how to avoid CIU equivalence and show directly contextual equivalence from OGS bisimilarity. The key observation is to change the initial configuration of terms and contexts, reusing a construction already appearing in the literature. It has been used as a proof technique for the genericity lemma in Takahashi's paper [Tak94]. Because of this technique previously appearing in Takahashi's work and that we only need to change the initial game composition, the talk will refer to the technique in OGS as *Takahashi's Gambit*.

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| **Previous Initial Conf.** | $\langle \epsilon \mid \epsilon \mid t \rangle$ | $\mid$ | $\langle \epsilon \mid D :: \epsilon \rangle$ |
| **Takahashi's Gambit Conf.** | $\langle [\alpha \leftarrow t] \cdot \epsilon \mid \epsilon \rangle$ | $\mid$ | $\langle \epsilon \mid \epsilon \mid D[\alpha] \rangle$ |

Switching these initial configurations does not change the result of composing them (they both evaluate $D[t]$), only switching the initial focus of the LTS on what should be evaluated first. In an evaluation context, the term in the hole can be evaluated first (following the previous soundness constructions), but in an arbitrary context, the term plugged in the hole may even never be evaluated in the composition $D[t]$, which is why Takahashi's gambit configurations do not focus on the plugged-in term but start by evaluating the context.

Further details about closed/open terms and call-by-value operational semantics (that usually constrain the OGS environment to be entirely made of values) will hopefully be addressed during the talk. They have been addressed in call-by-value variants of Takahashi's genericity reasoning [AL24, Lan25]. The key point is that $D[t]$ is equivalent (in a strong rewriting sense) to $D[\alpha] \mid [\alpha \leftarrow t]$ and also equivalent to $D[\alpha\, \mathtt{unit}] \mid [\alpha \leftarrow \lambda\_.t]$.

**Future and Related Work.** By switching the initial configurations, we also change what OGS bisimilary on terms $t \sim_{OGS} u$ typically means. We are currently investigating how to prove that the notion of equivalence on terms is unchanged, which formally might require an *up-to separation* technique [KLT22].

To the best of our knowledge, OGS soundness proofs all mention some variant of CIU equivalence, which is why we would like to present this alternative proof sketch.

Borthelle's PhD thesis includes a formalization of OGS in the Rocq proof assistant, including soundness for *substitution equivalence* [Bor25, Definition 4.13], which is a substitution-based variant of CIU equivalence. Lancelot's PhD thesis also includes a *substitution equivalence/preorder* [Lan25, Definition 1.6.15], which is directly related to contextual equivalence (not a CIU restriction) via Takahashi's trick: it is used in a formal development in the Abella proof assistant [LAV25] and intended to be the formal counterpart of Takahashi's proof technique for genericity.

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