Seeking the primitive categorical structure of concurrent game semantics

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Context. Games and strategies may be organized as various categorical models depending on the target language. Of course, they can form a cartesian closed category for interpreting call-by-name [HO00, AJM00]. But there are also models of call-by-value, which can be organized as a closed Freyd category [HY99] or a cartesian closed category with a strong monad [AM97]; and many others.

Concurrent games in particular support a similar variety, also accommodating linear languages. For call-by-name, they naturally form a so-called *relative Seely category* [Cla24], a slight generalization of Seely categories handling the semantics of a programming language with both linear and non-linear function types. In particular, it has a cartesian closed Kleisli category. For call-by-value, in the linear setting they can organized into a linear Freyd category [CdVW19].

Typically, all these categorical structures are achieved by providing concrete strategies for all structural morphisms. Sometimes this is a bit unwieldy, like the definition of the premonoidal tensor in [CdVW19]. Almost always there is a significant repetition of work; for instance many laws look like the neutrality of copycat for composition, without being *exactly* that. It feels like there should be a cleaner algebraic story, and this is the problem that this work aims to address.

Objectives. In this proposed talk, we will present work in progress towards a "primitive" categorical structure for concurrent game semantics. More precisely, we are seeking a categorical axiomatization with the following characteristics:

- It should be *minimal* and *elementary*.
- It should naturally fit concurrent game semantics: it should be simple to construct this structure concretely, and all the involved constructions on games and strategies should be simple.
- It should be *general*: one should be able to derive, just from this basic categorical structures, as many as possible of the familiar categorical structures from categorical semantics: linear and non-linear version of call-by-name, call-by-value, and call-by-push-value. These derived structures should be compatible with their pre-existing concrete counterparts in the literature.
- It should be *functorial*: there should be matching notions of morphisms between models, lifting between structure-preserving functors for the various derived structures.

- Finally, it should be *compatible* with other concrete denotational models such as variations and categorifications of the relational model, non-idempotent intersection types etc; in the sense that these relational models should be instances, and there should be structure-preserving functors to these.

This work is *not* about finding the "right" general notion of linear call-by-push value [CFM16], although this could be a way to study that question through the lense of game semantics. Rather, it is to provide a toolbox making it easy to reason about various evaluation strategies within (concurrent) game semantics.

Contributions. Our proposition is to start with a categorical structure matching the category of negative concurrent games. Technically, this would be simply a cartesian dialogue category. Drawing inspiration from recent unpublished work by Melliès, it can also be equipped with an exponential modality, which takes the form of a comonad relative to an appropriate negation functor.

In this talk, we will describe the matching concrete structure in the category of negative thin concurrent games. We will show how to derive from this axiomatization various other categorical models of interest in denotational semantics. For call-by-name, we will show how to obtain a relative Seely category and hence a cartesian closed category. For call-by-value, we will show how to obtain a closed Freyd category along with its linear variant. Interestingly, it turns out that these models of call-by-value can be equivalently obtained in two different ways, mimicking the "positive games" [HY99] and "family construction" [AM97] approches to game semantics models of call-by-value.

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