

Concurrency in linear and non-linear game semantics

Hugo Paquet, Inria & ENS-PSL



Based on joint work with
Flavien Breuvar (LIPN) and Philip Saville (U. Sussex)



Fundamental law of sequential and parallel composition

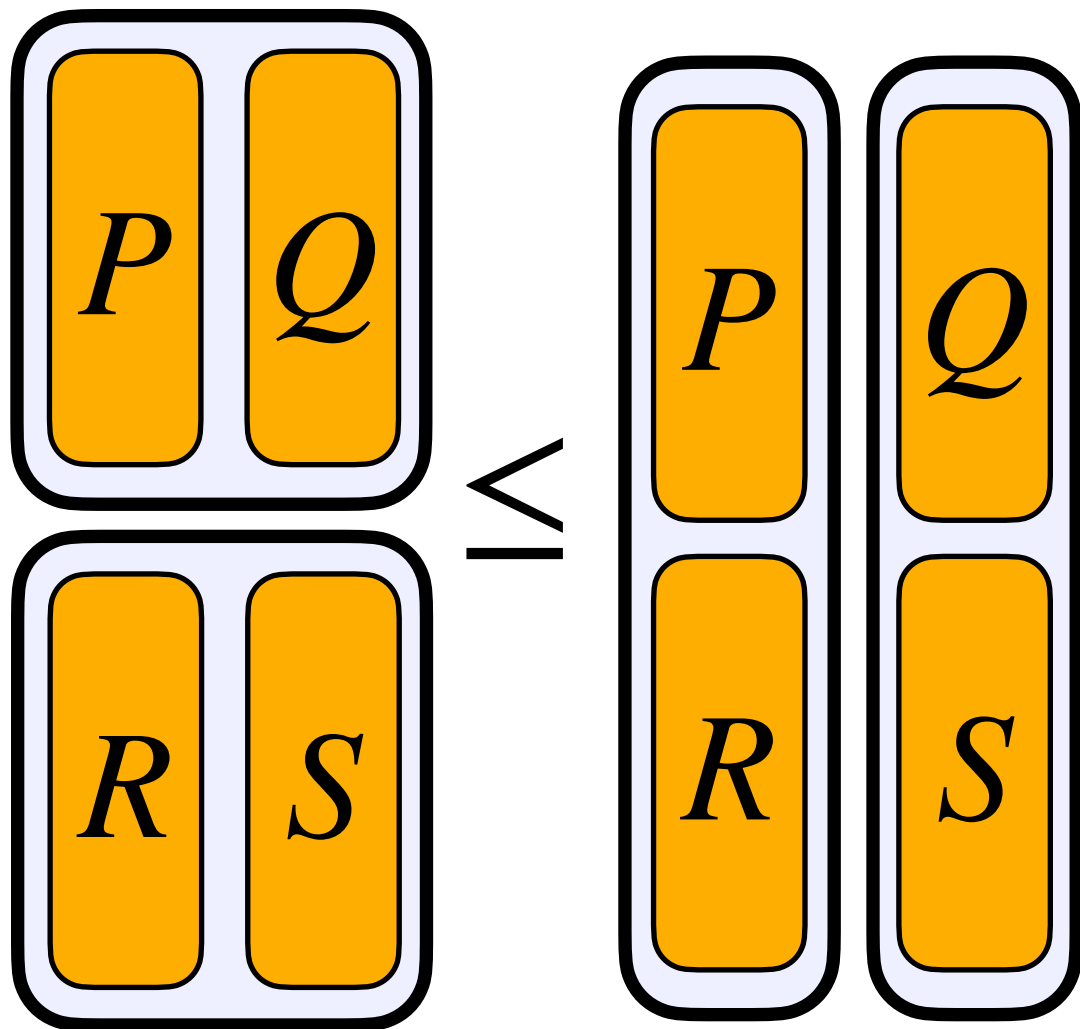
cf. Hoare 2013

$$\begin{aligned} (P \parallel Q) ; (R \parallel S) \\ \leq (P ; R) \parallel (Q ; S) \end{aligned}$$

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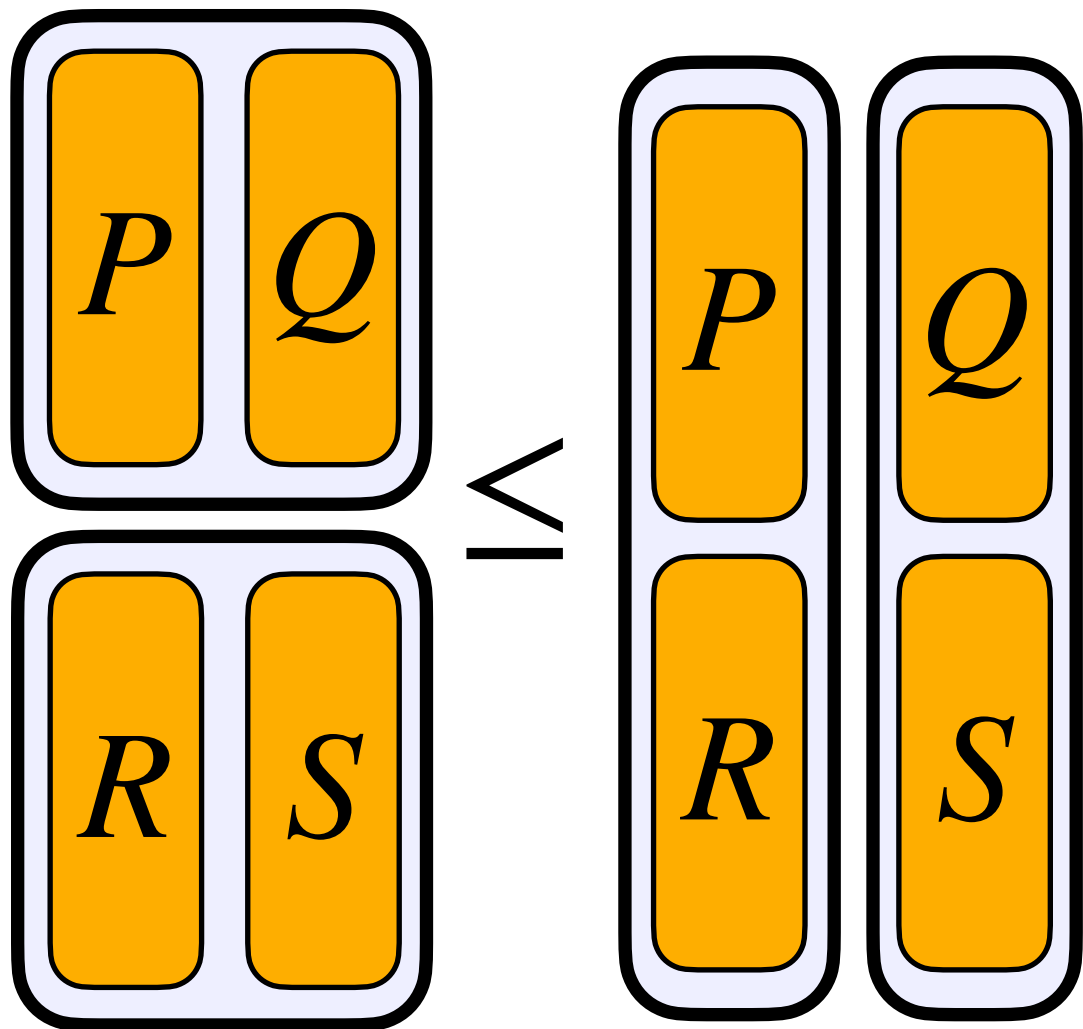
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Fundamental law of sequential and parallel composition

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Goal: connect this with various semantic models.

- Monads
- Game semantics
- Linear / non-linear logic

Concrete example: pomsets

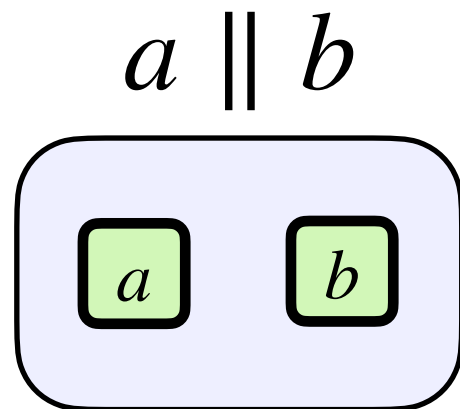
Simple processes built from atomic events: $\boxed{a}$, $\boxed{b}$, ...
and parallel and sequential composition.

Model: pomsets (finite labelled posets).

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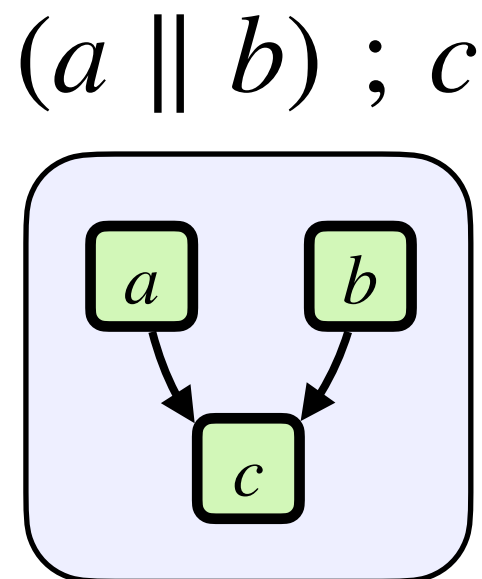
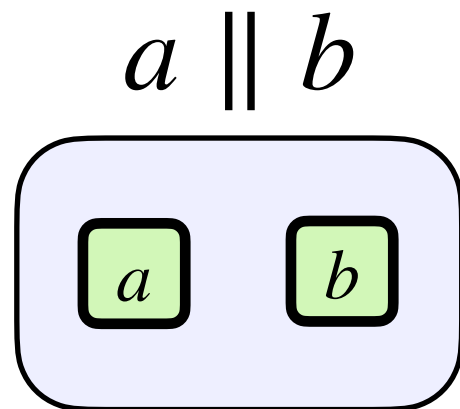
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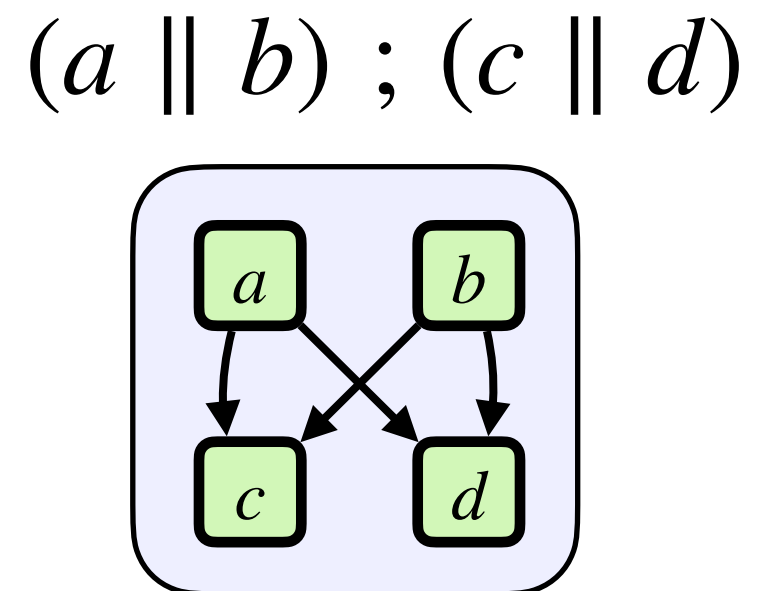
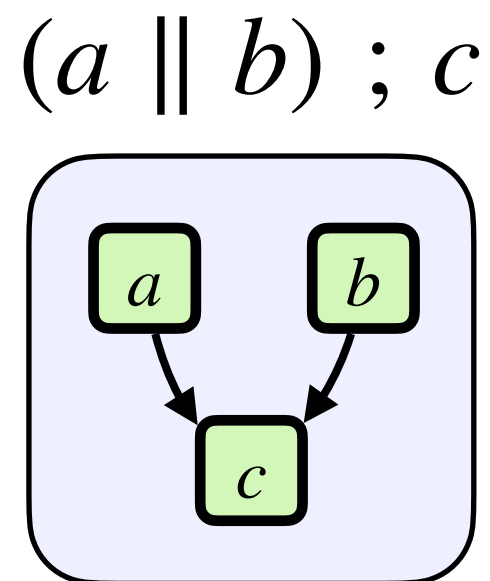
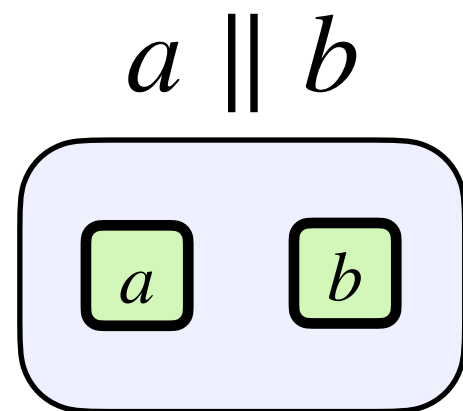
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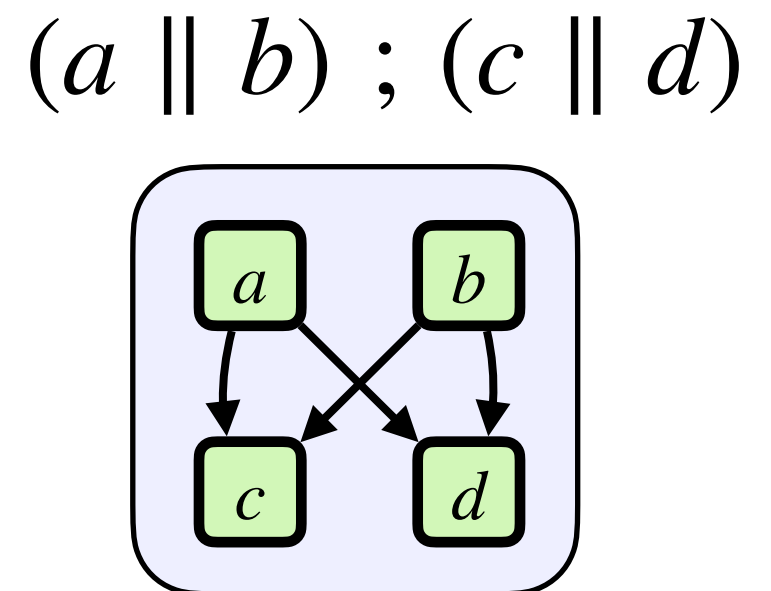
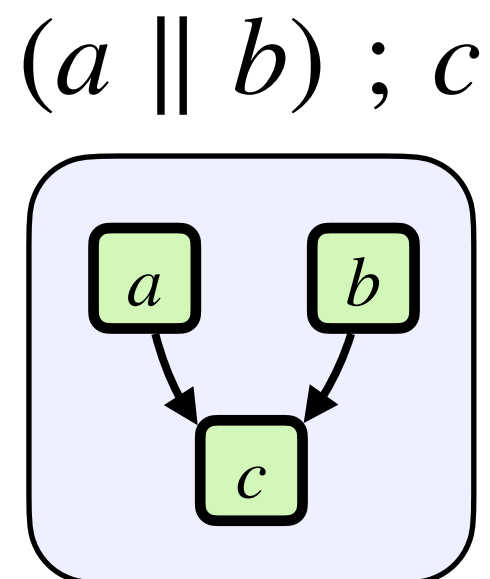
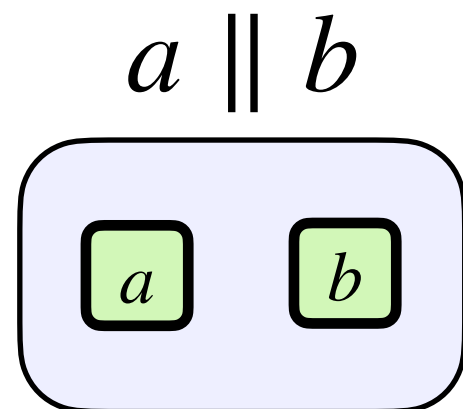
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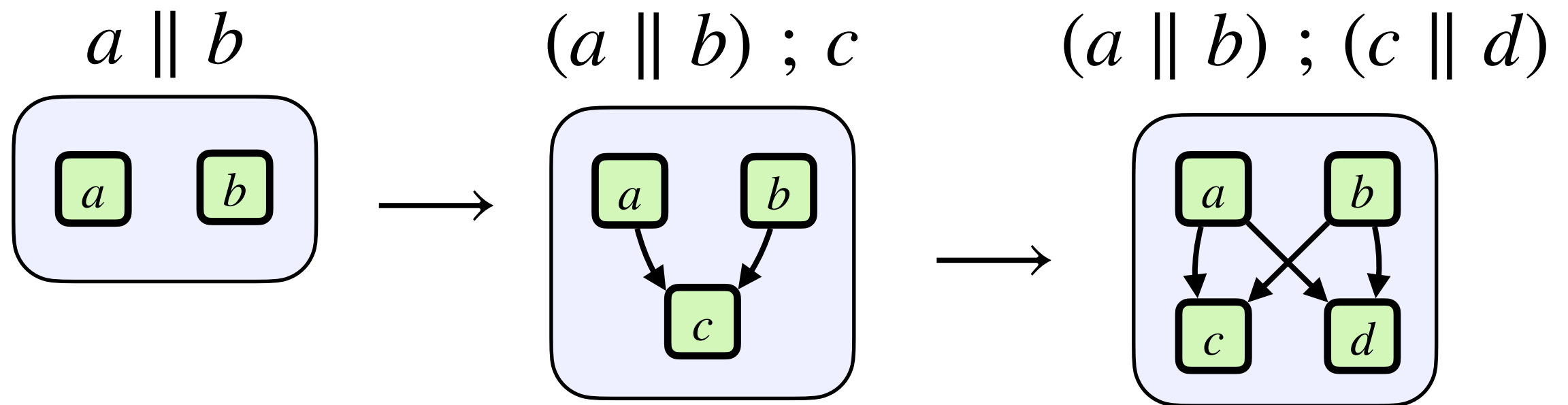
Possible traces:

$\{a, b, ab, ba, abc, bac\}$

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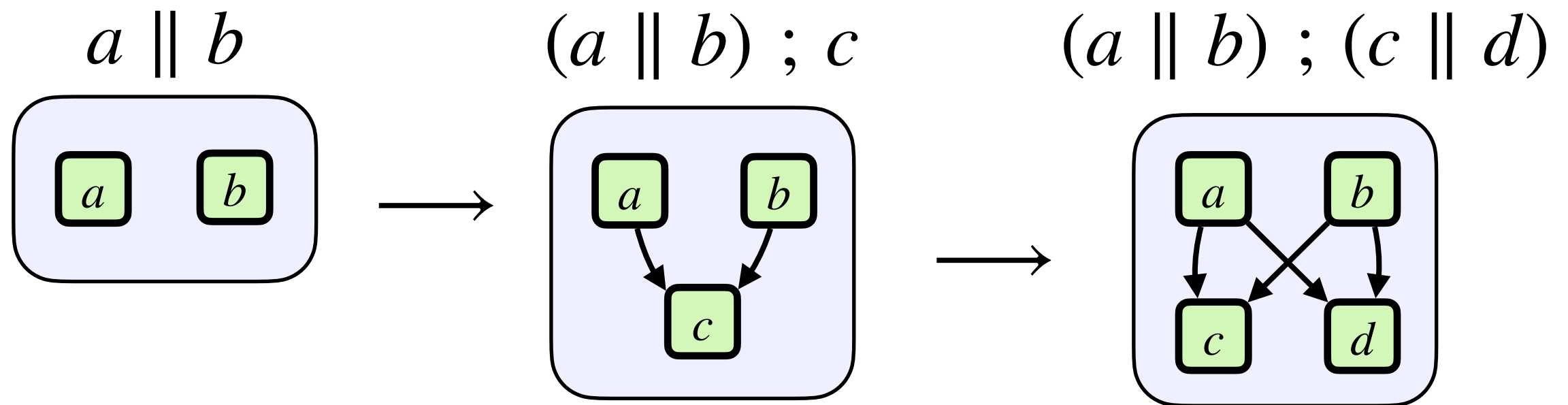
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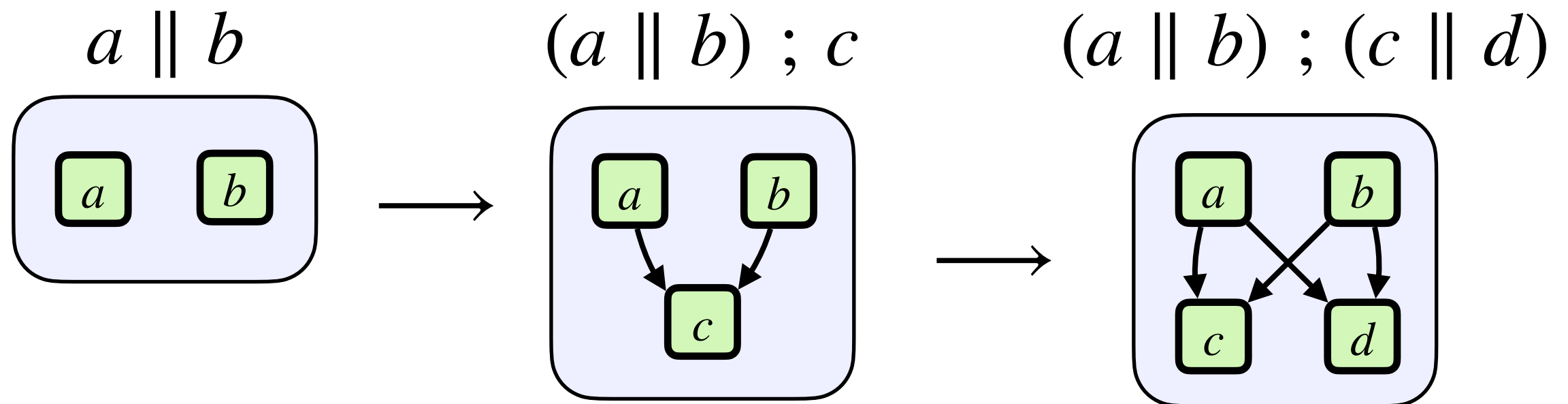


Morphisms of pomsets: functions preserving sets of traces.

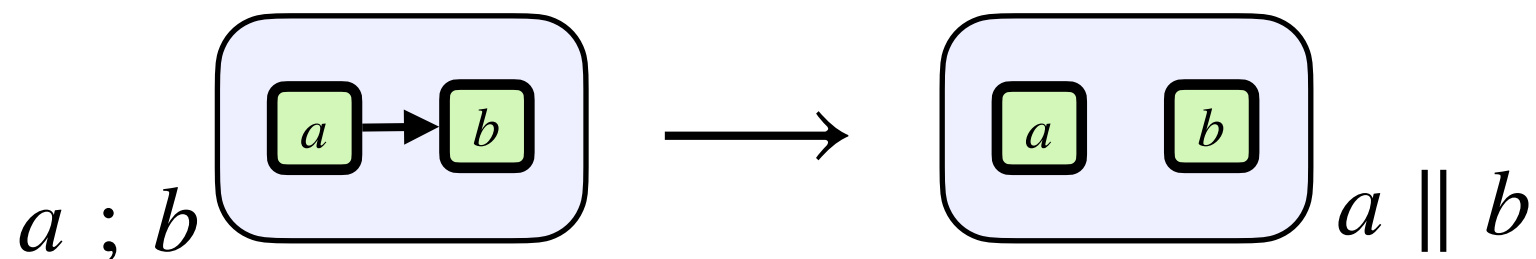
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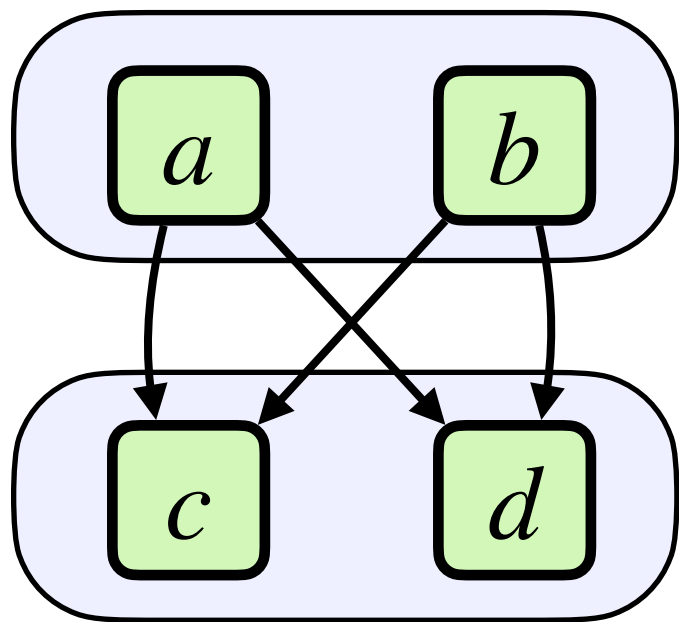
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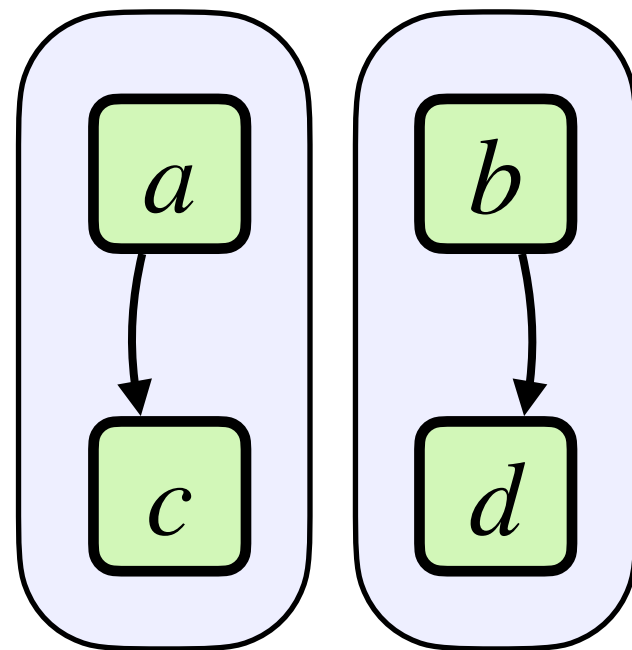
Concrete example: pomsets

Fundamental law as weak interchange:

$$(a \parallel b) ; (c \parallel d)$$



$$(a ; c) \parallel (b ; d)$$



More generally: a family of maps

$$(P \parallel Q) ; (R \parallel S) \longrightarrow (P ; R) \parallel (Q ; S)$$

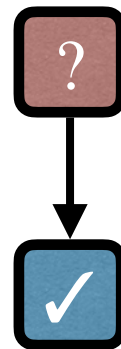
forming a natural transformation.

Concurrency in game semantics

Now events are either Player  or Opponent .

*E.g. a game
for simple processes.*

Proc

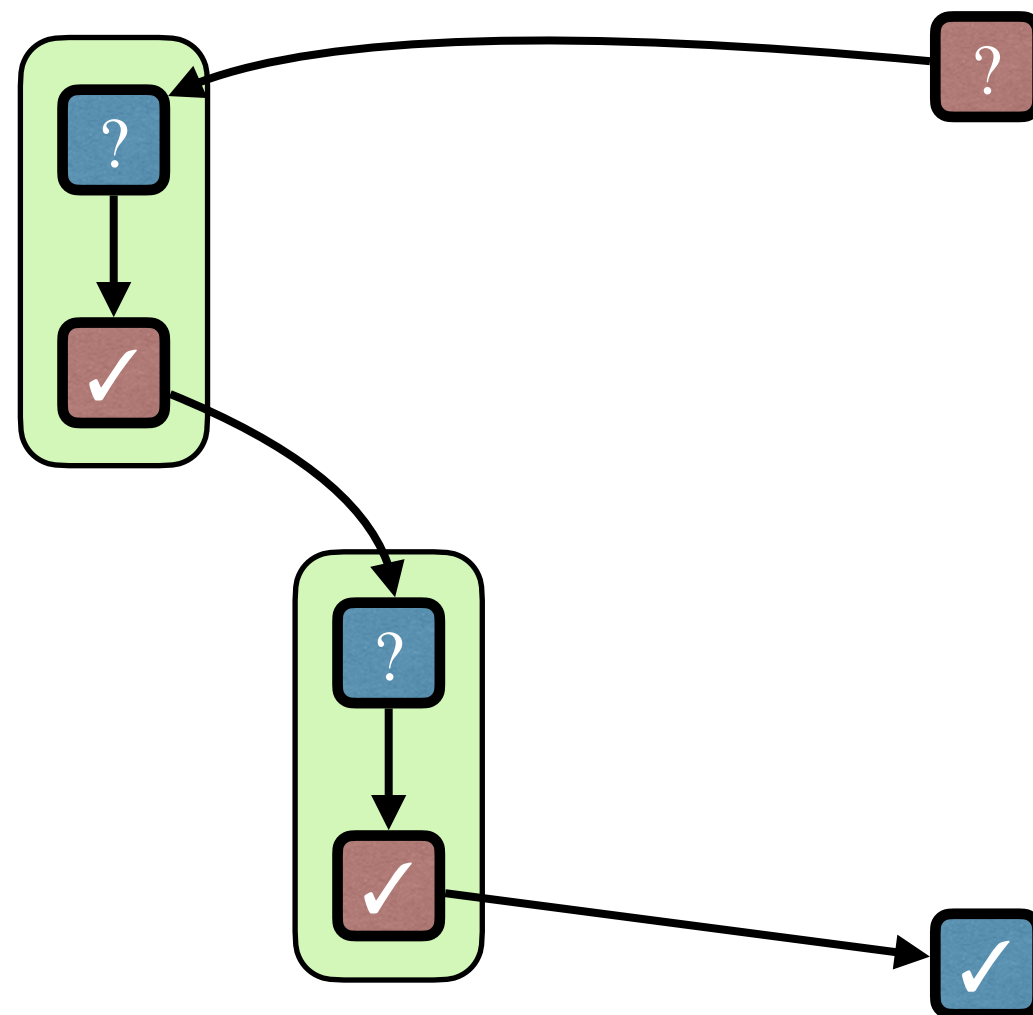


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$$\text{Proc} \otimes \text{Proc} \longrightarrow \text{Proc}$$

A strategy for sequential composition.



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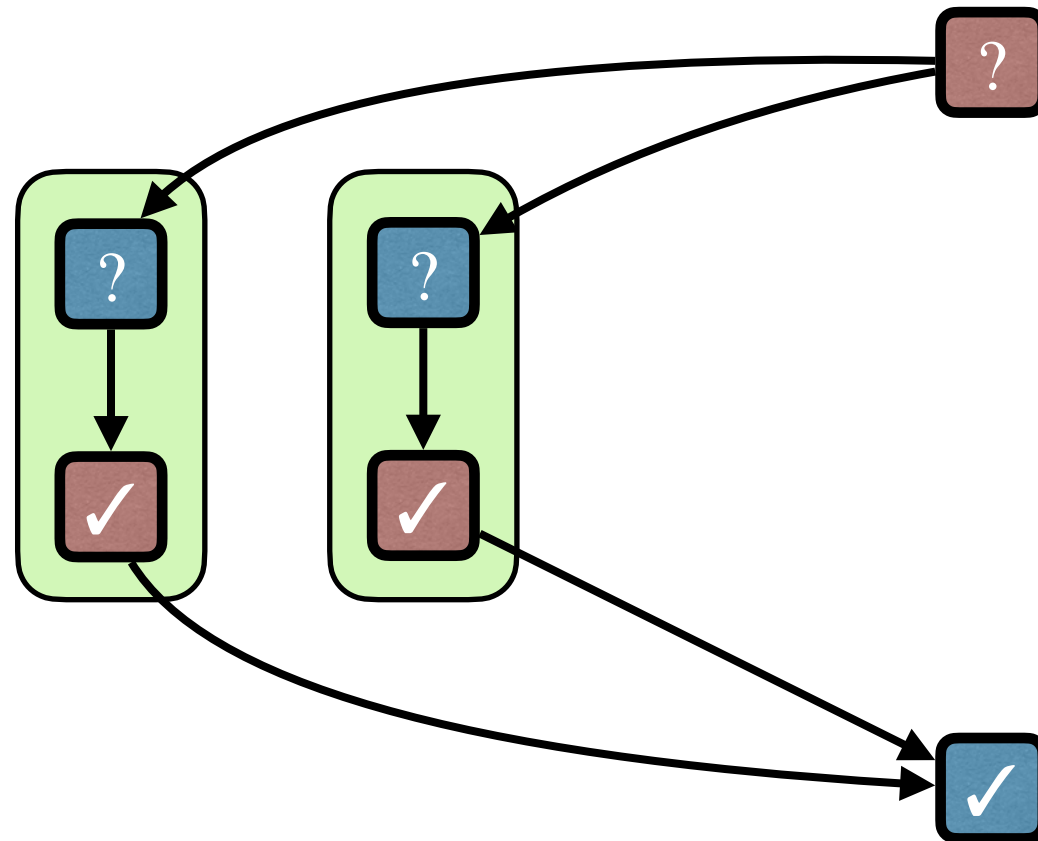
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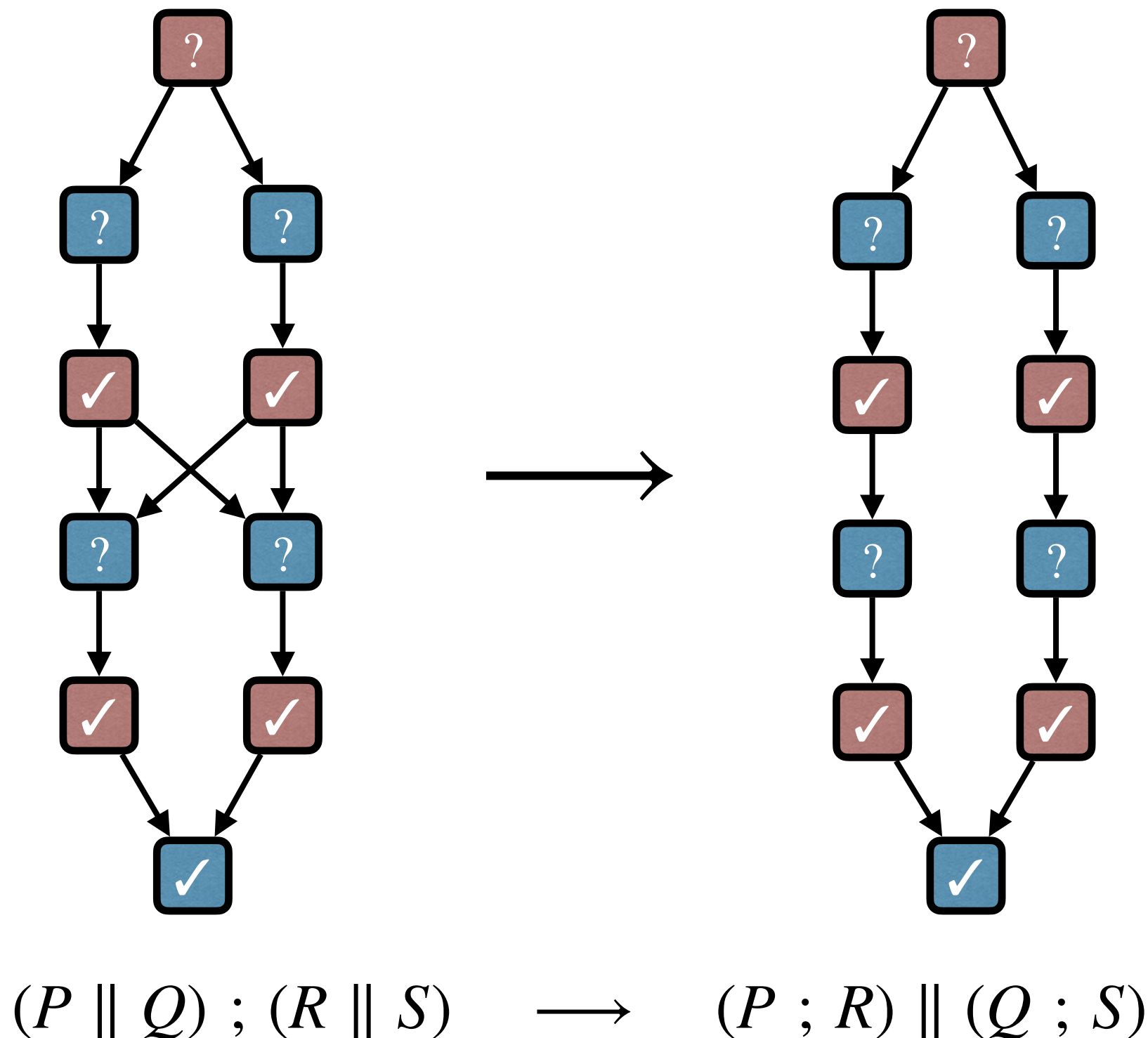
Concurrency in game semantics

The weak interchange law is represented by a map of strategies:

$$(P \parallel Q) ; (R \parallel S) \longrightarrow (P ; R) \parallel (Q ; S)$$

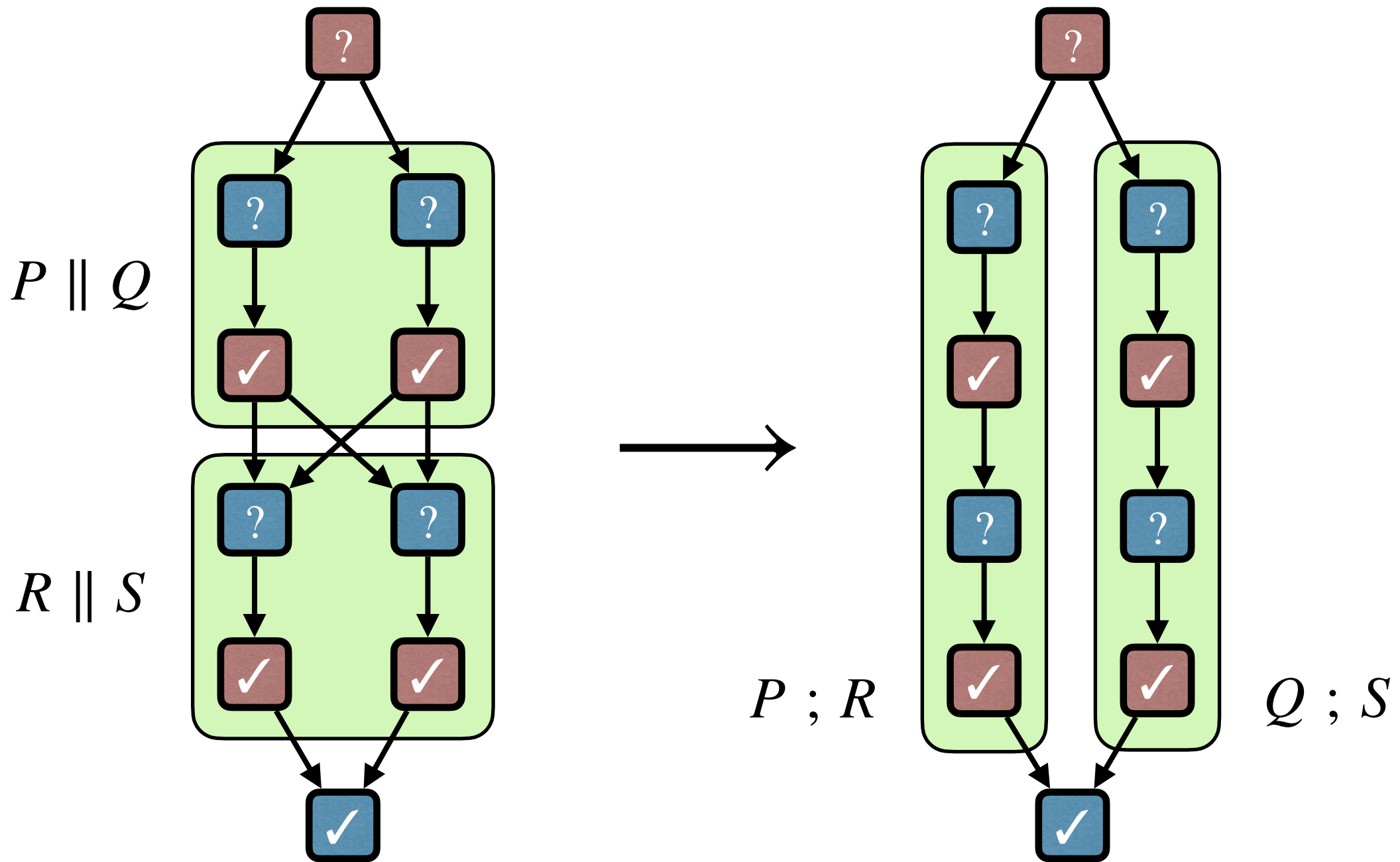
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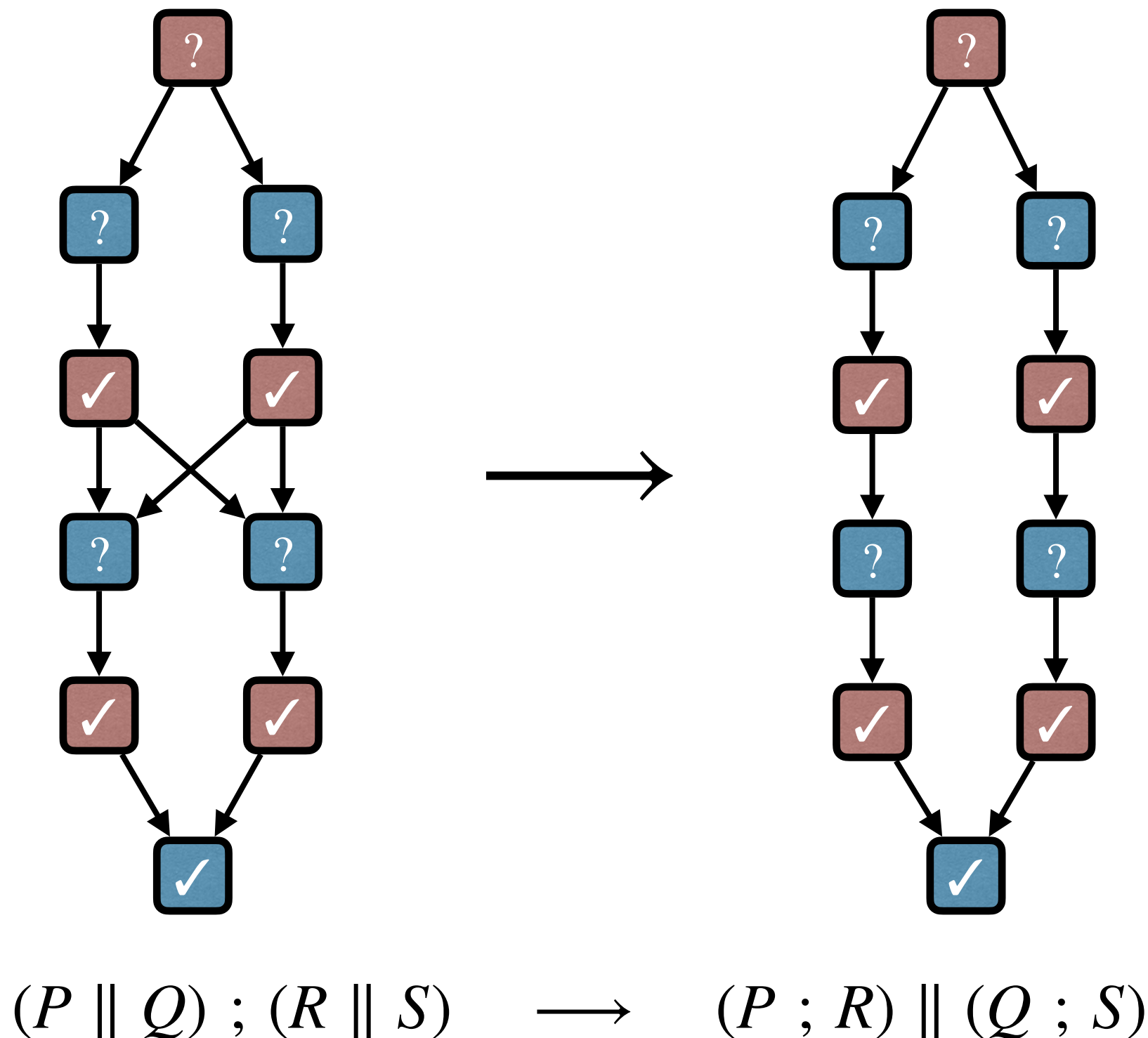
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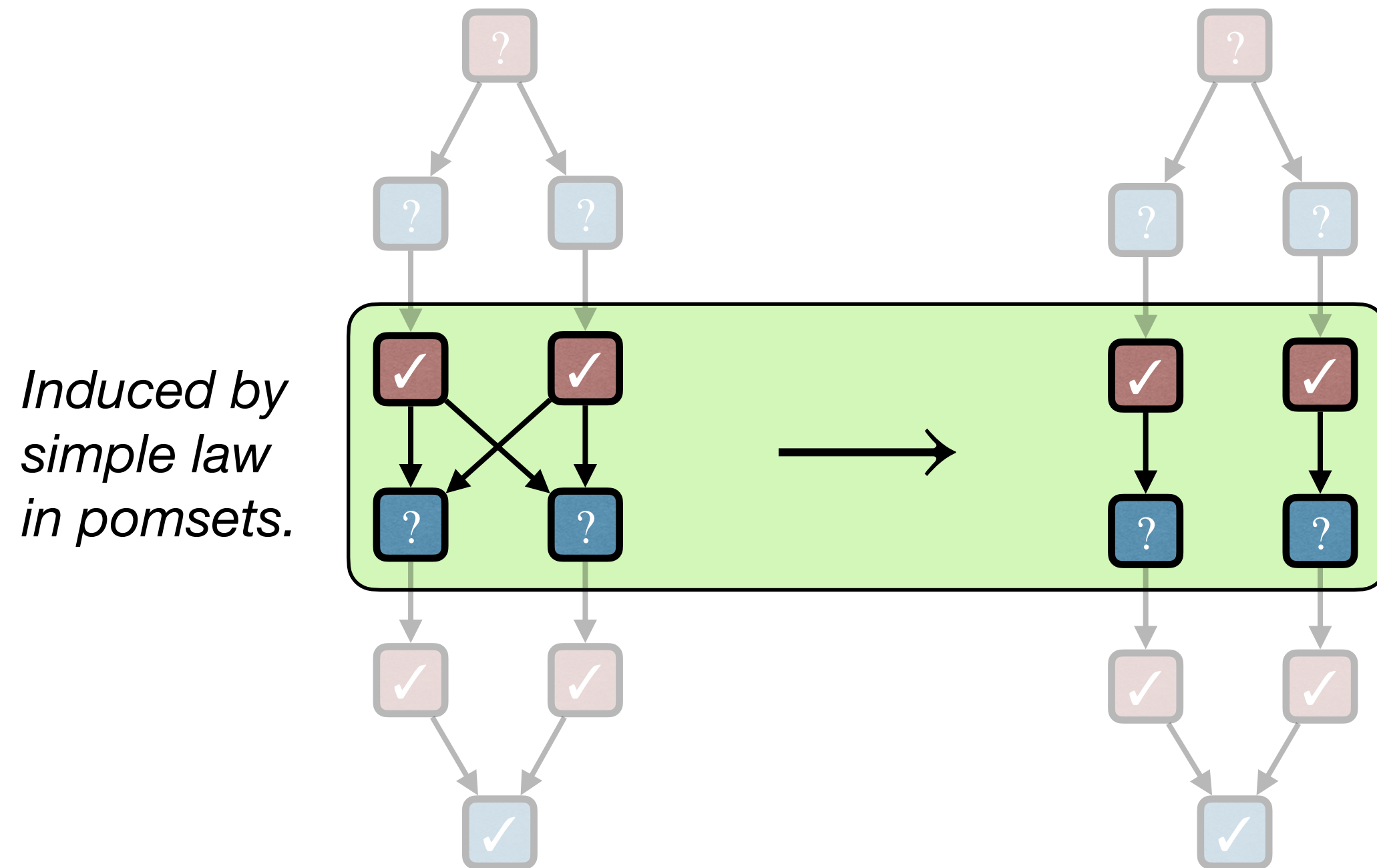
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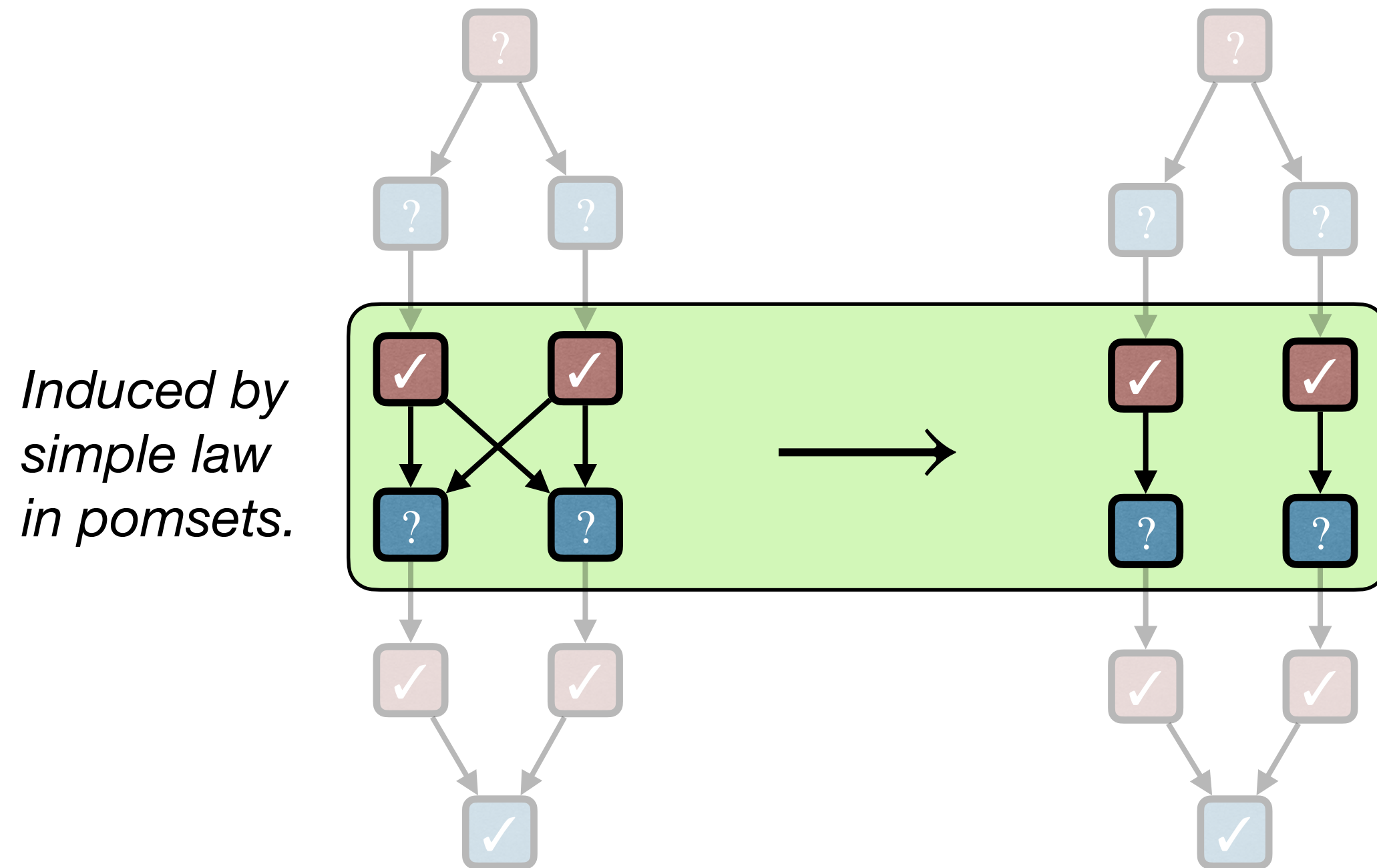
Concurrency in game semantics

The weak interchange law is represented by a map of strategies:



Concurrency in game semantics

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This is a **2-categorical** feature.

Concurrency and monads

Concurrency and monads

Rivas–Jaskelioff 2019, P–Saville 2024

Definition. A concurrent 2-monad (T, μ, η) is a “weakly” monoidal 2-monad:

$$\chi_{A,B} : TA \otimes TB \longrightarrow T(A \otimes B)$$

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Consequence: weak interchange law for Kleisli morphisms:

$$(M \parallel M') \circ (N \parallel N') \implies (M \circ N) \parallel (M' \circ N')$$

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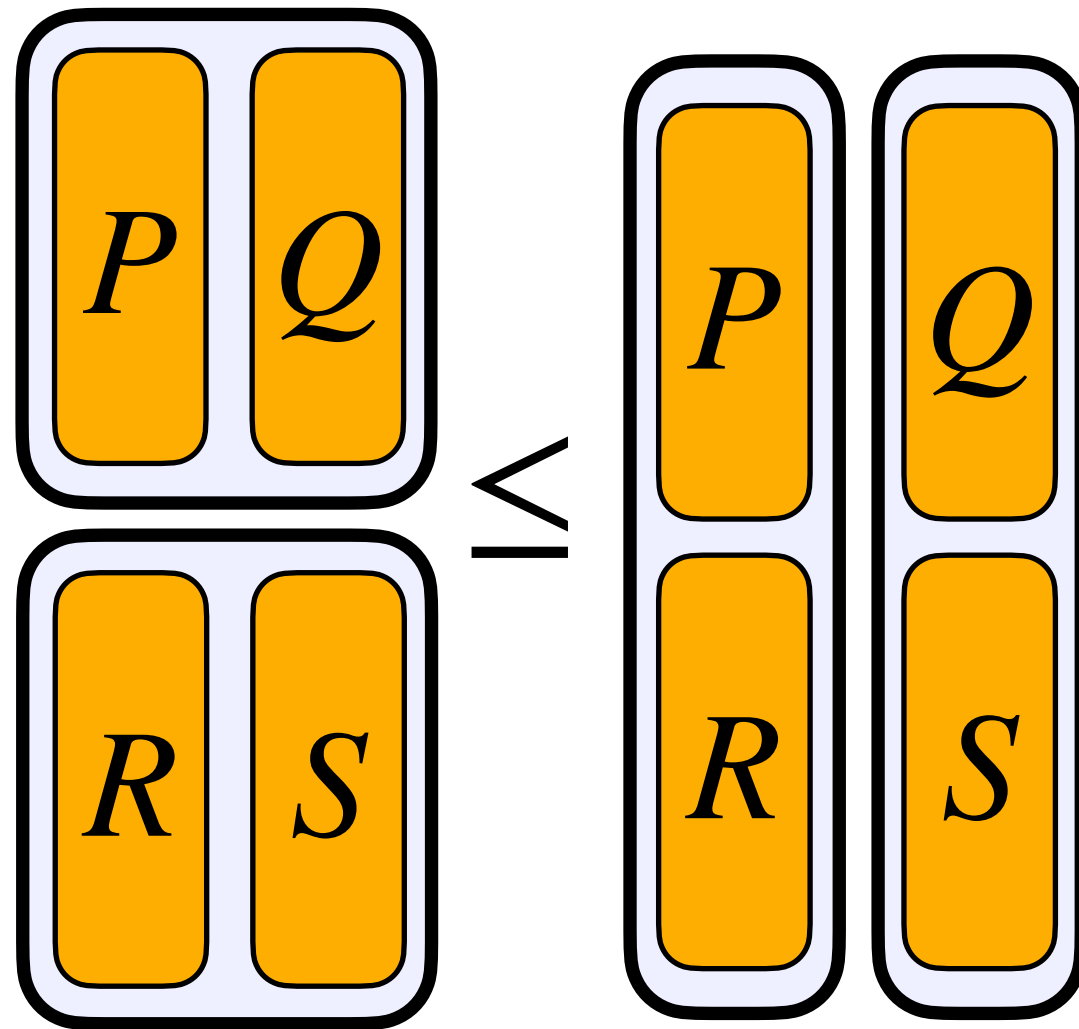
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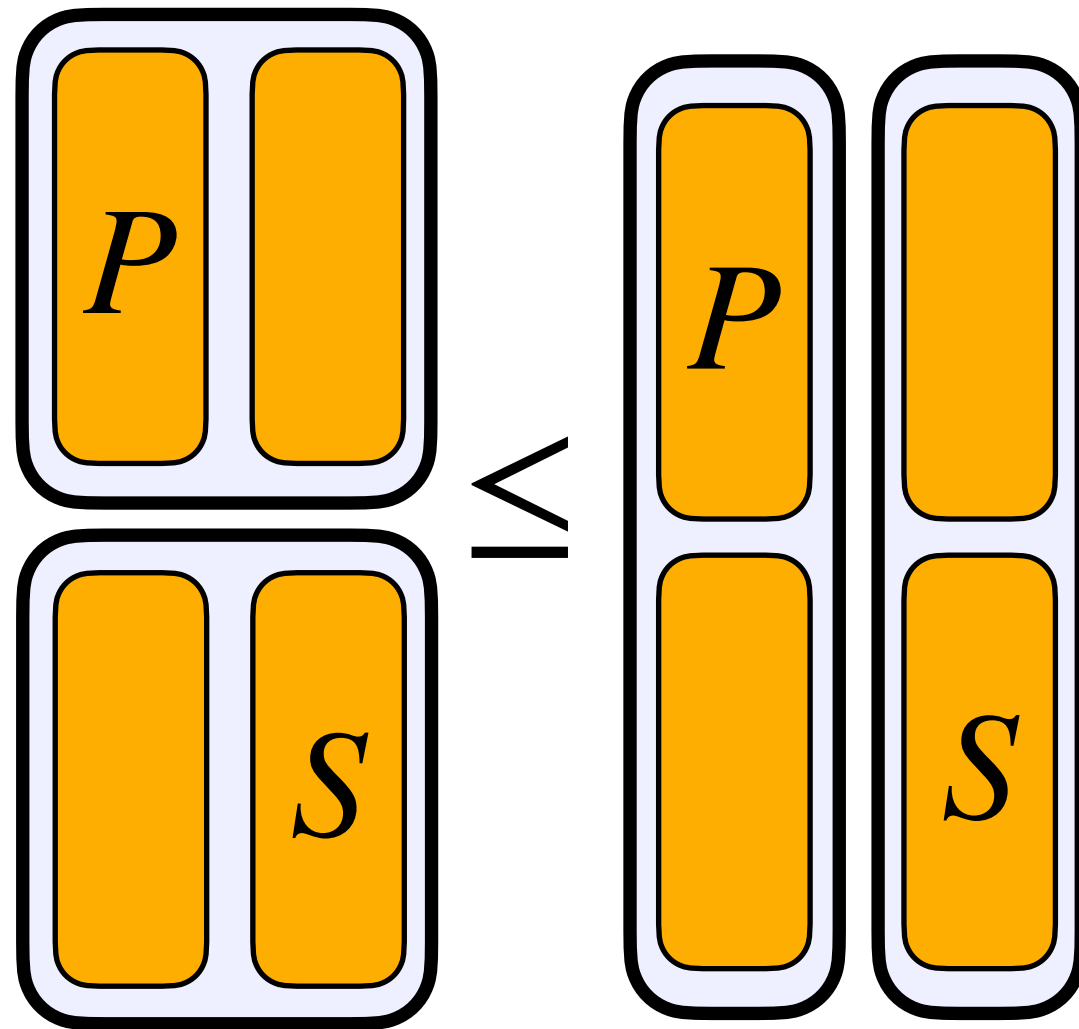
Concurrency and monads

Compare with commutativity:



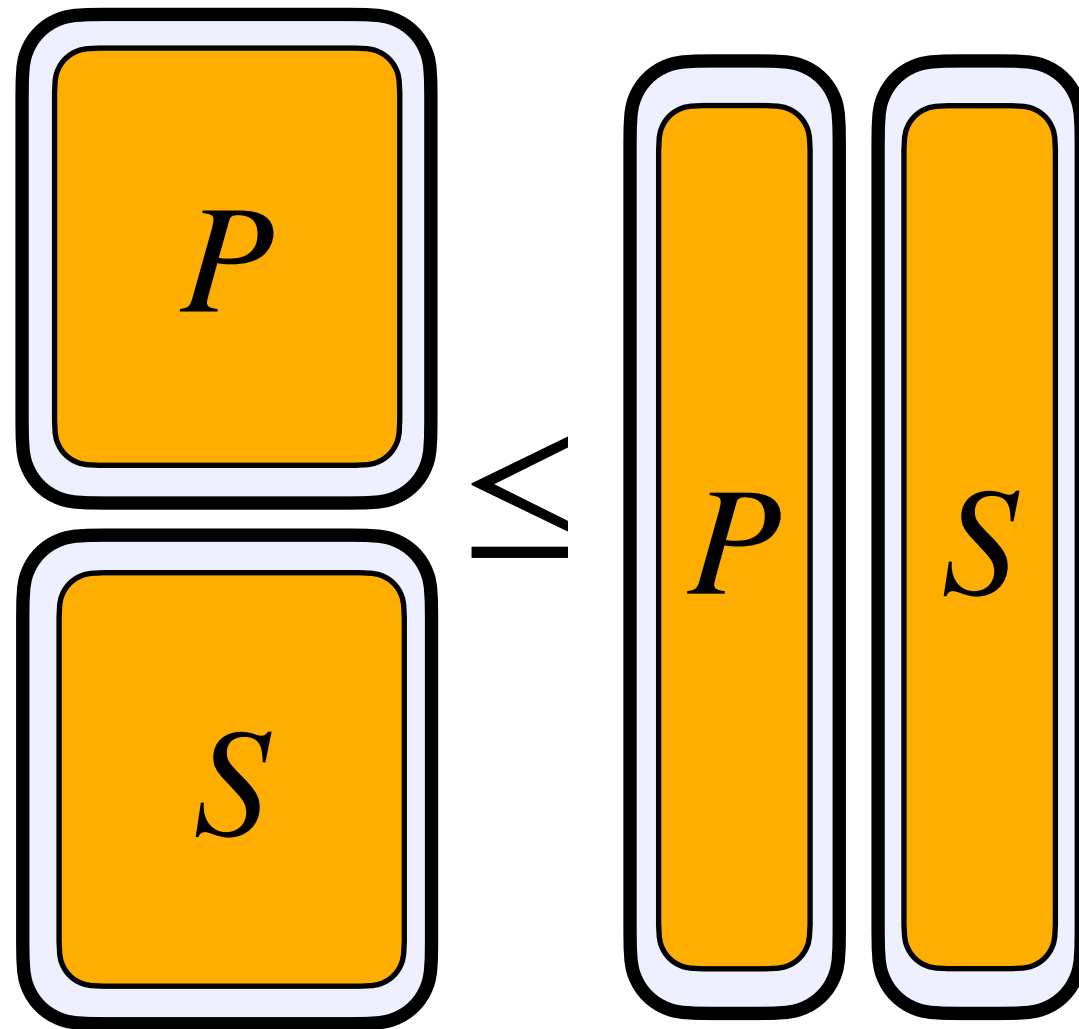
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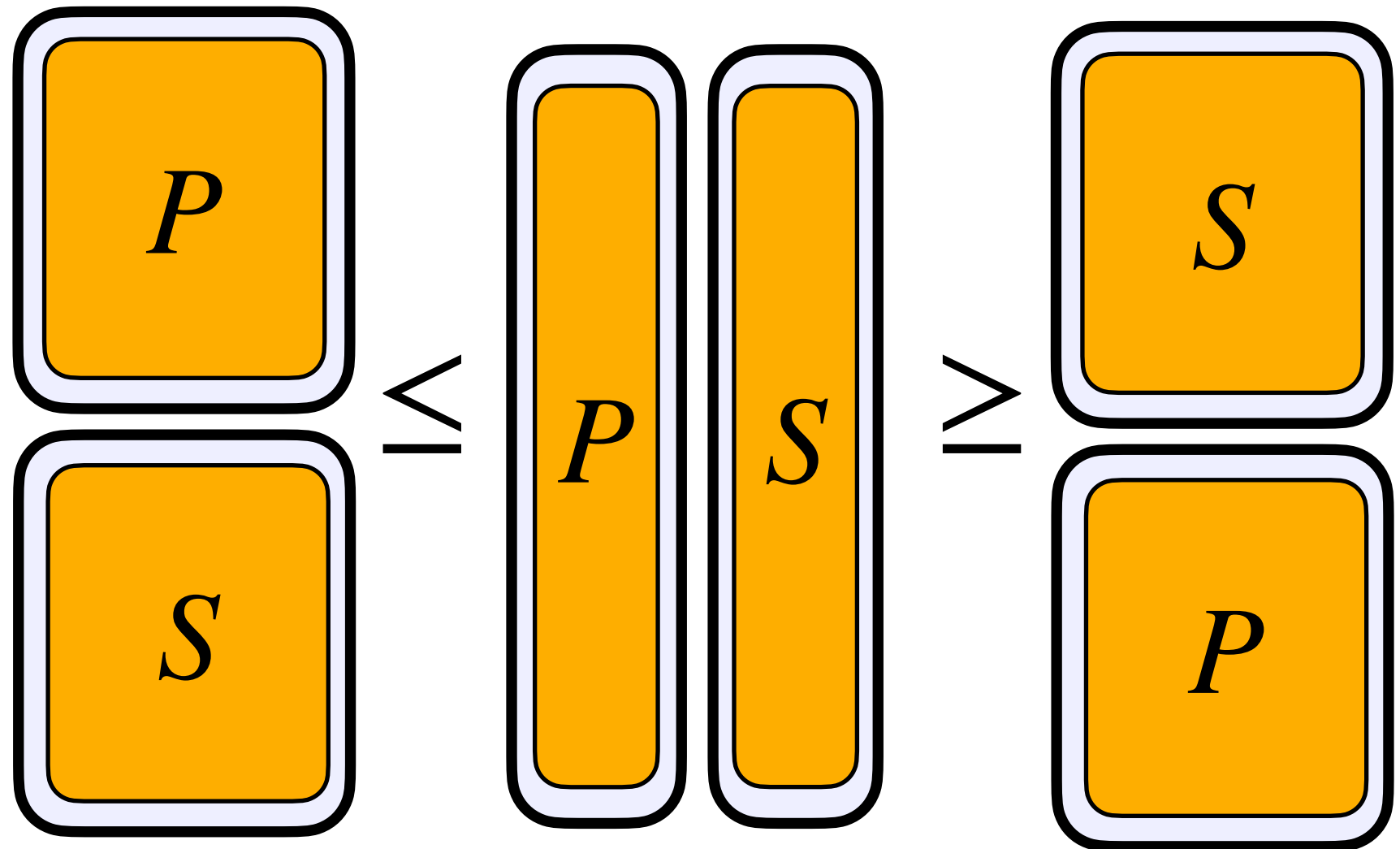
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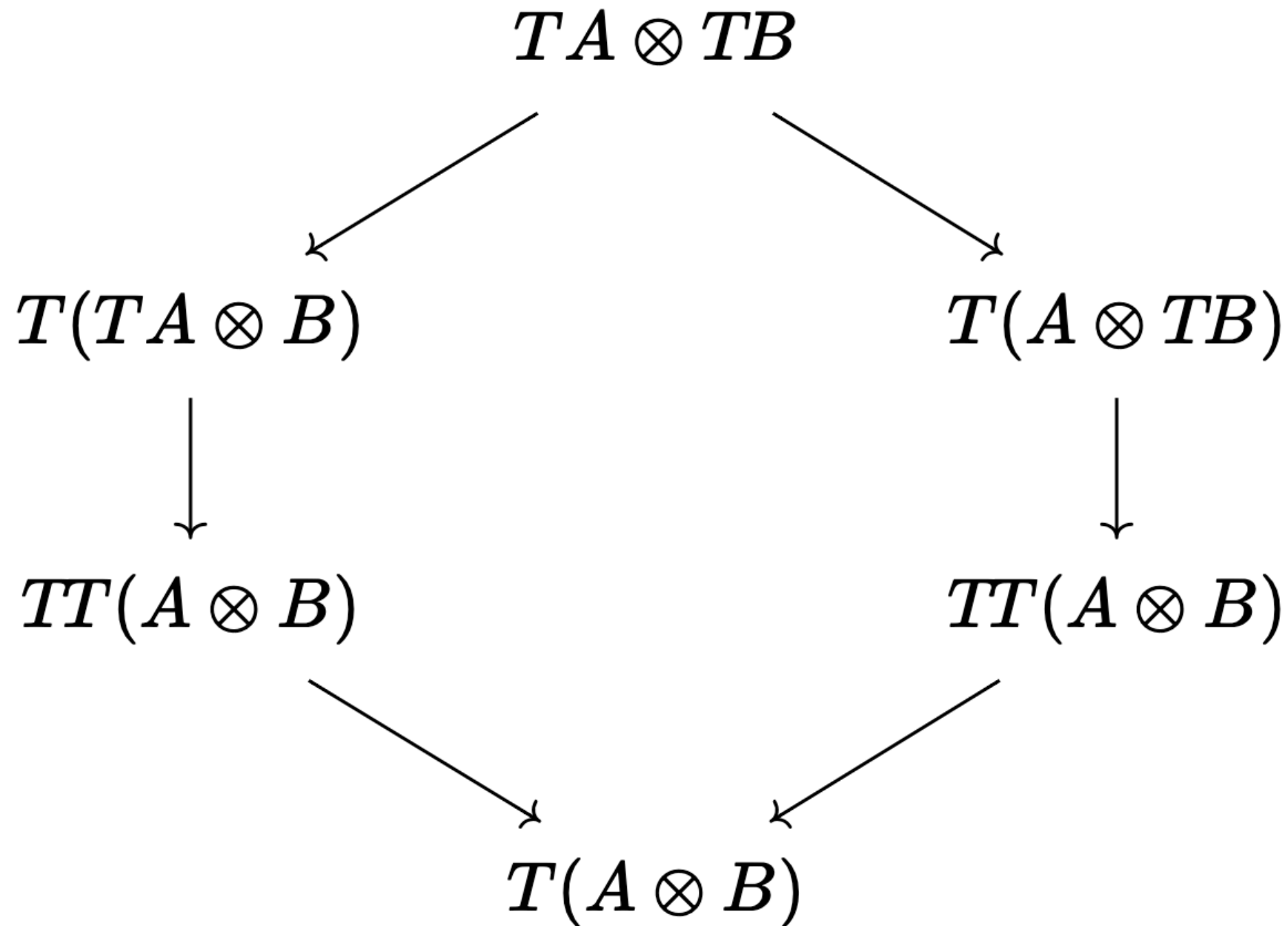
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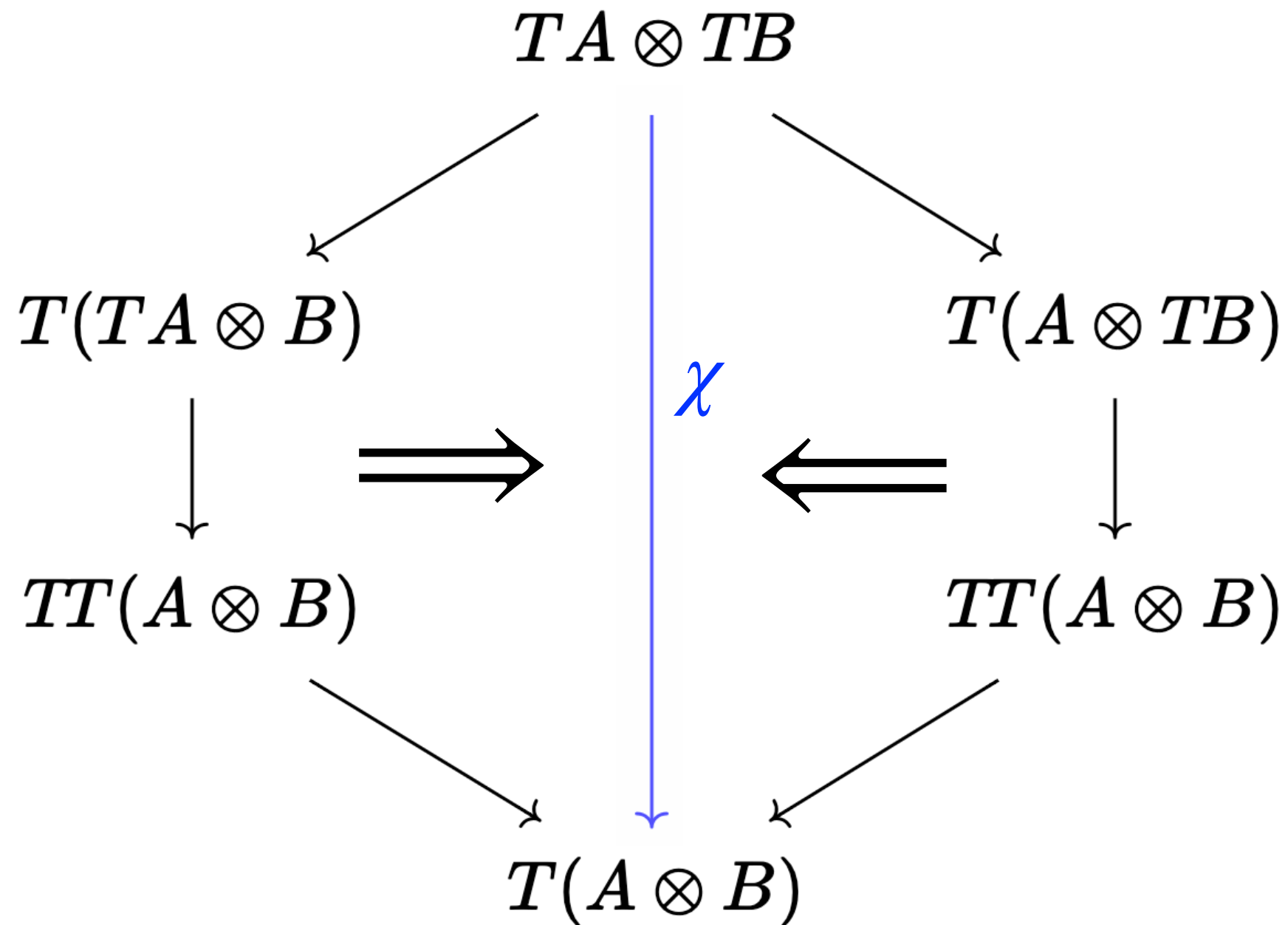
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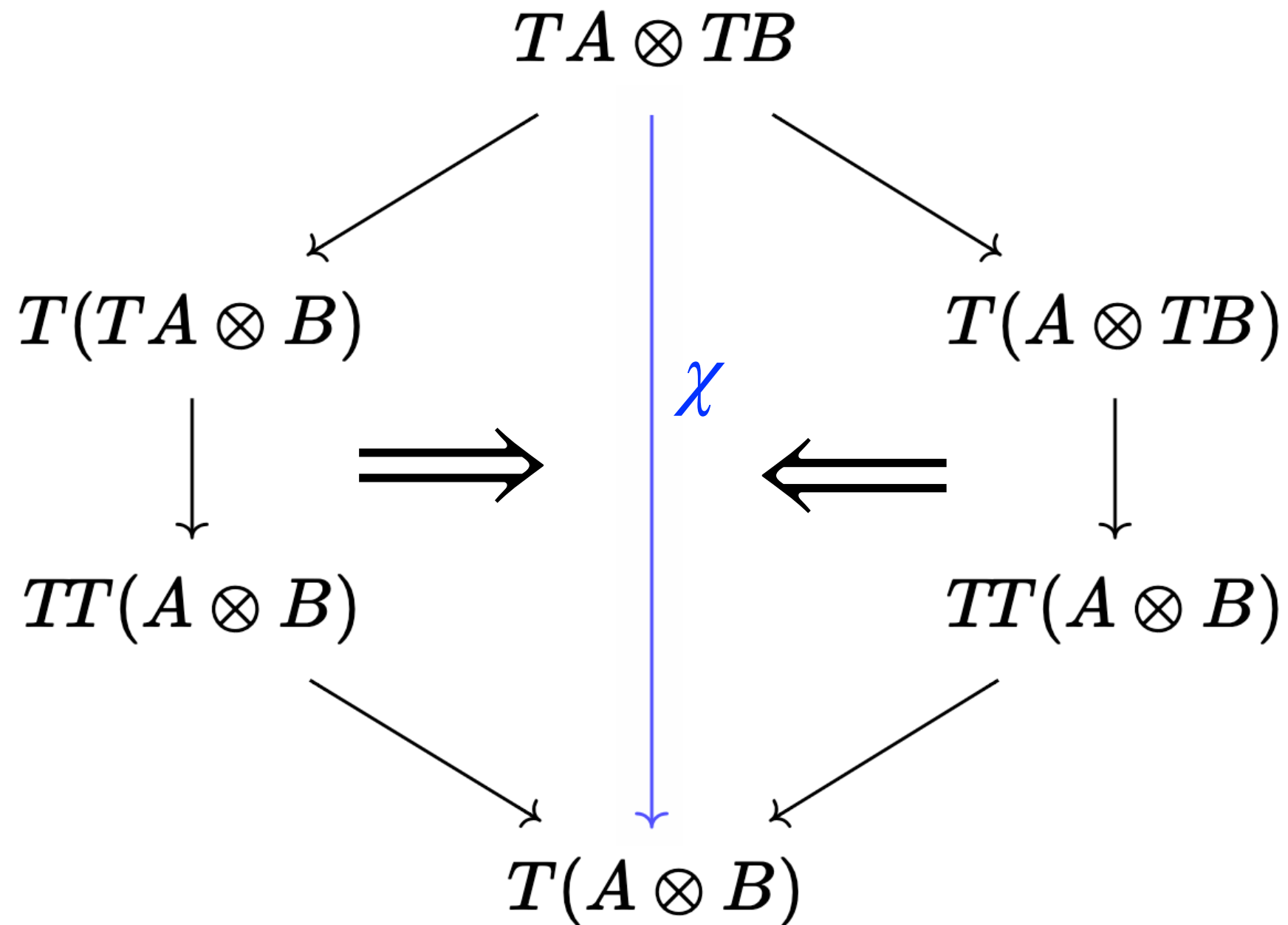
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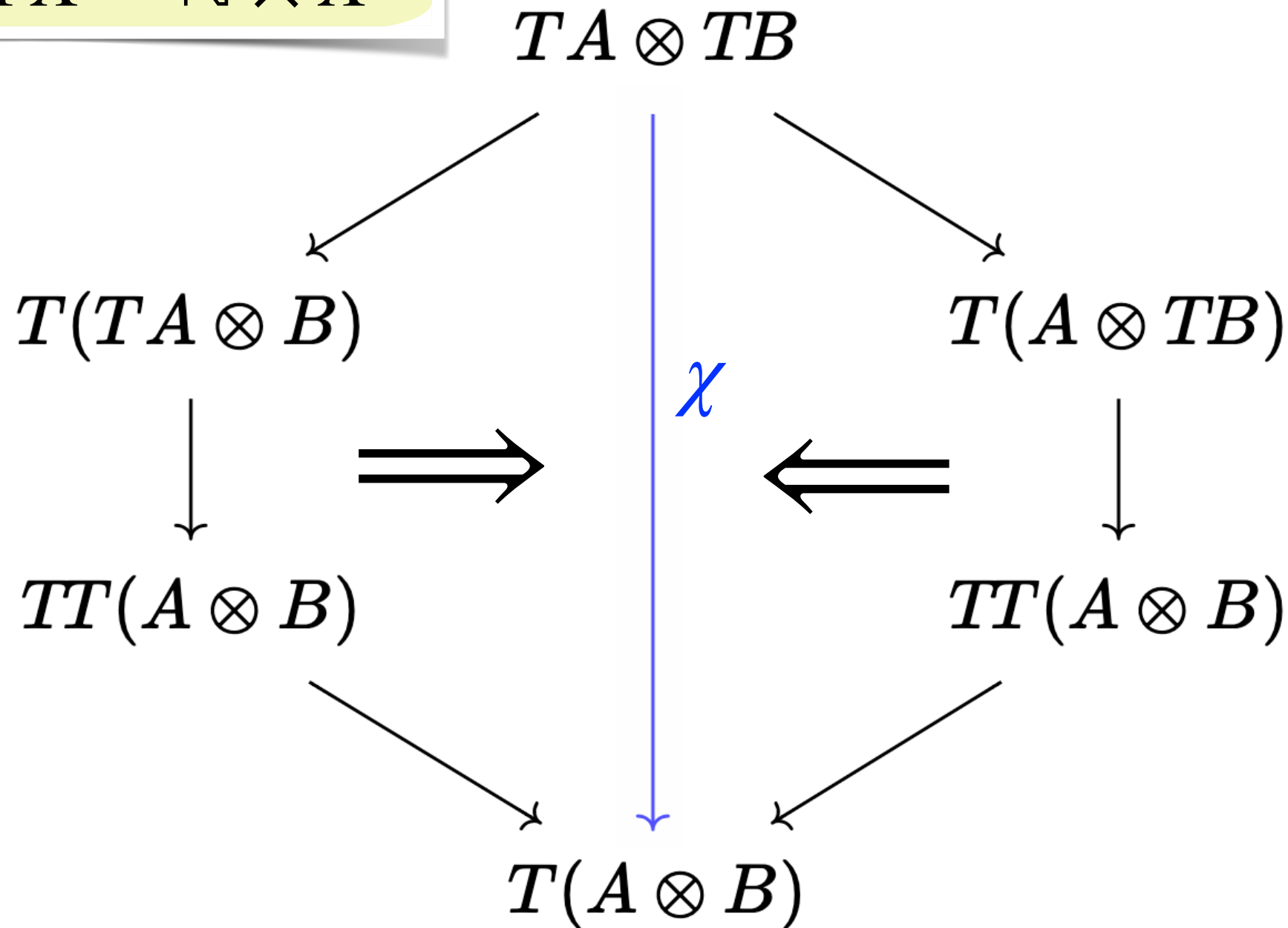
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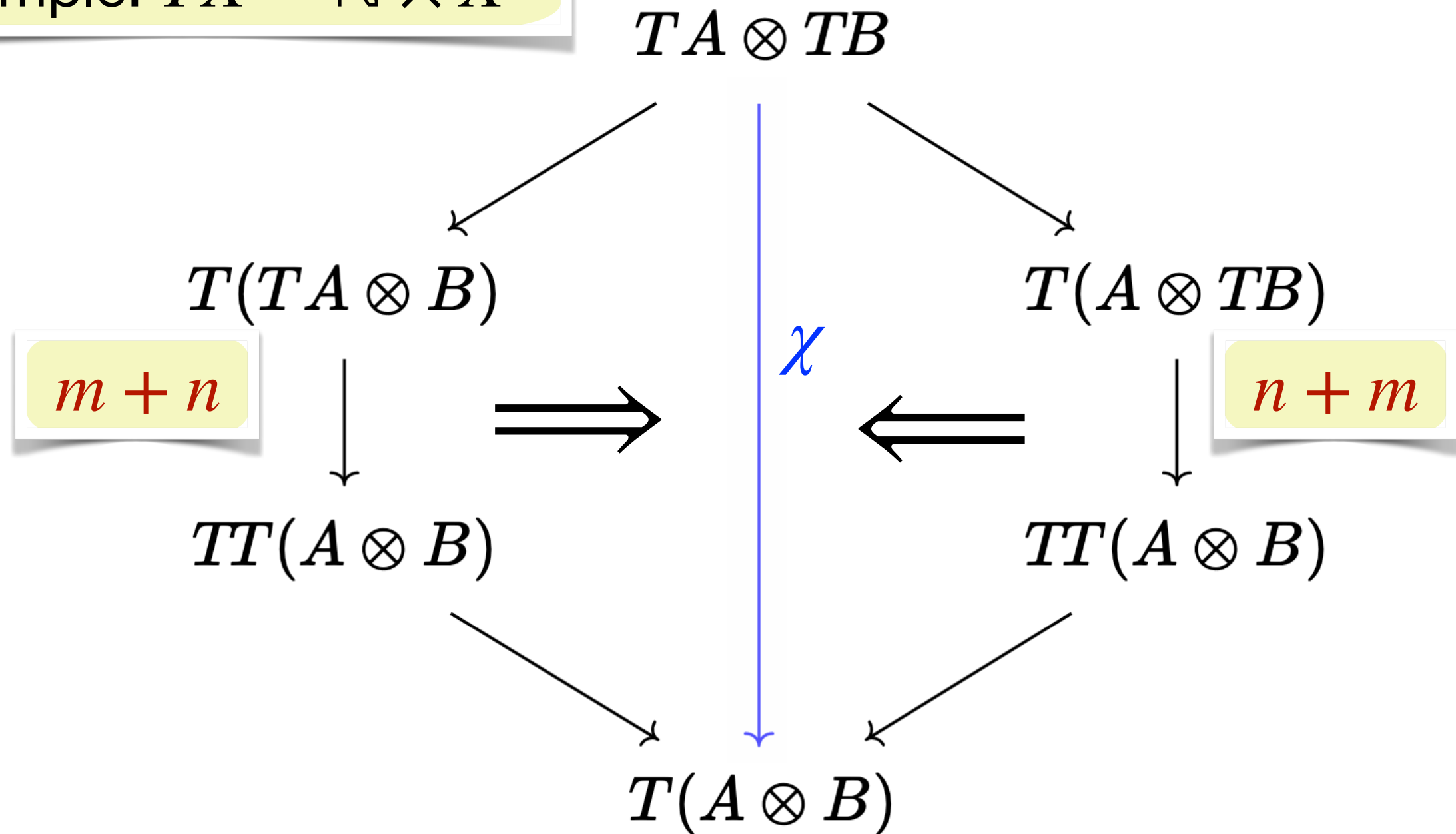
Example: $TX = \mathbb{N} \times X$



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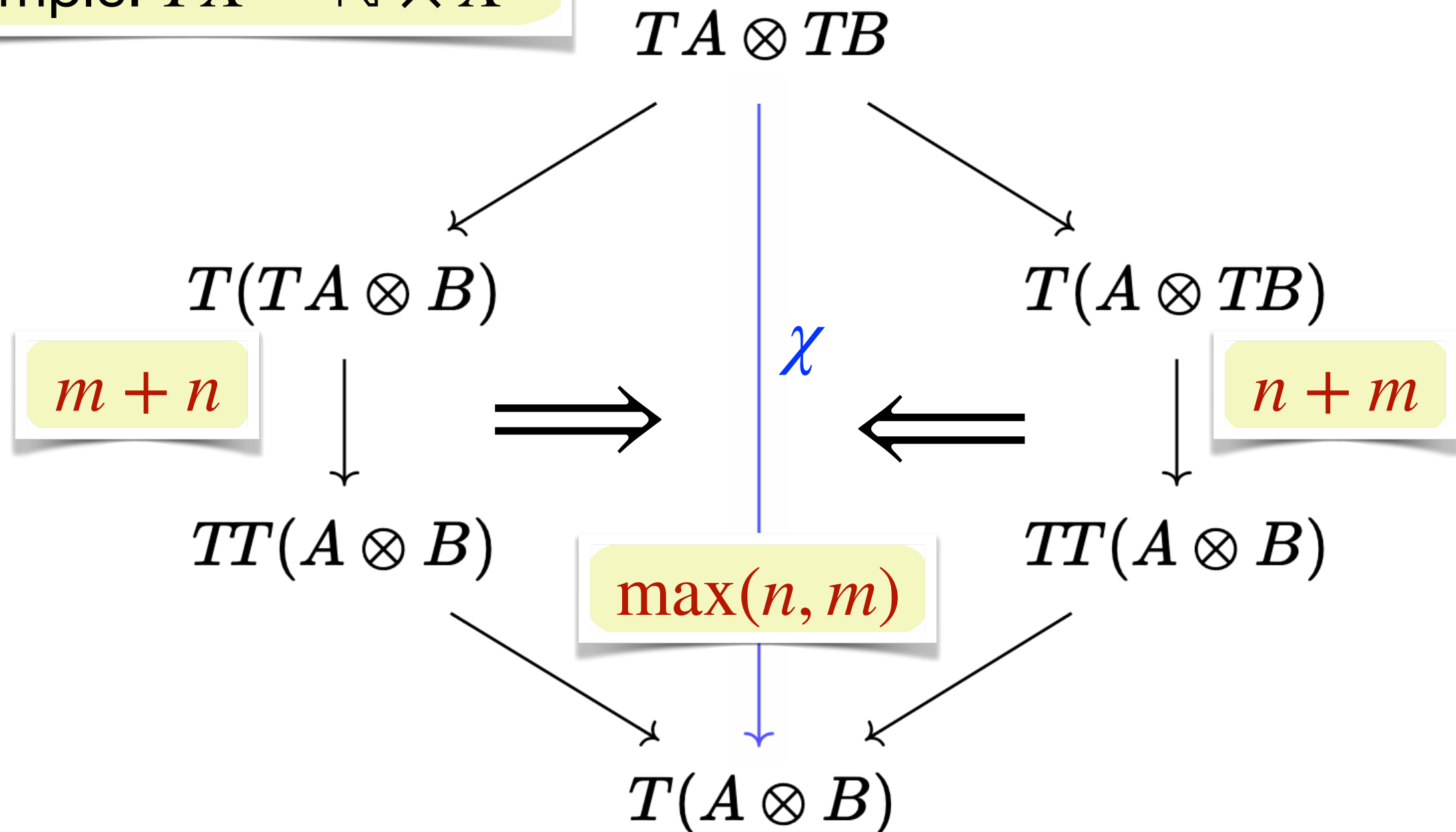
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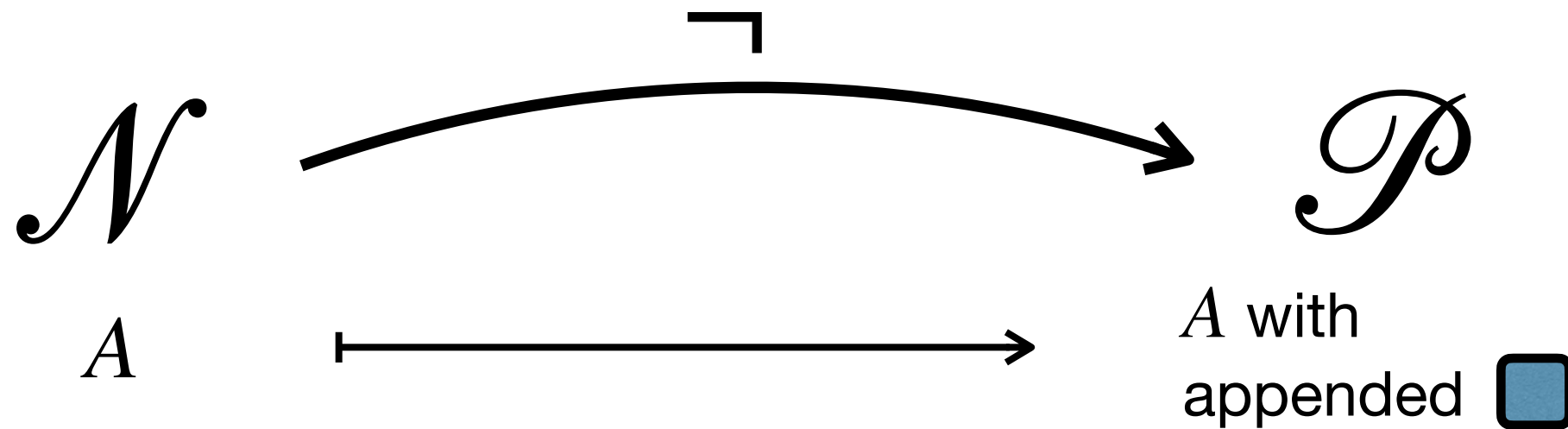
1. Examples, monads
- 2. Concurrency and dialogue structure**
3. Linear / non-linear games

Polarity and negation

- **Negative** games (Opponent  always starts)
and **positive** games (Player  always starts)

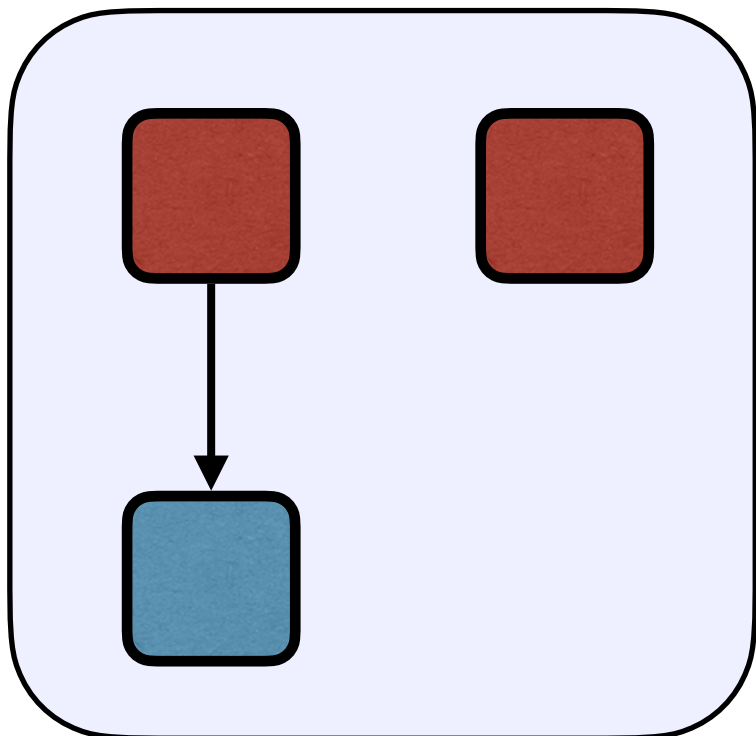
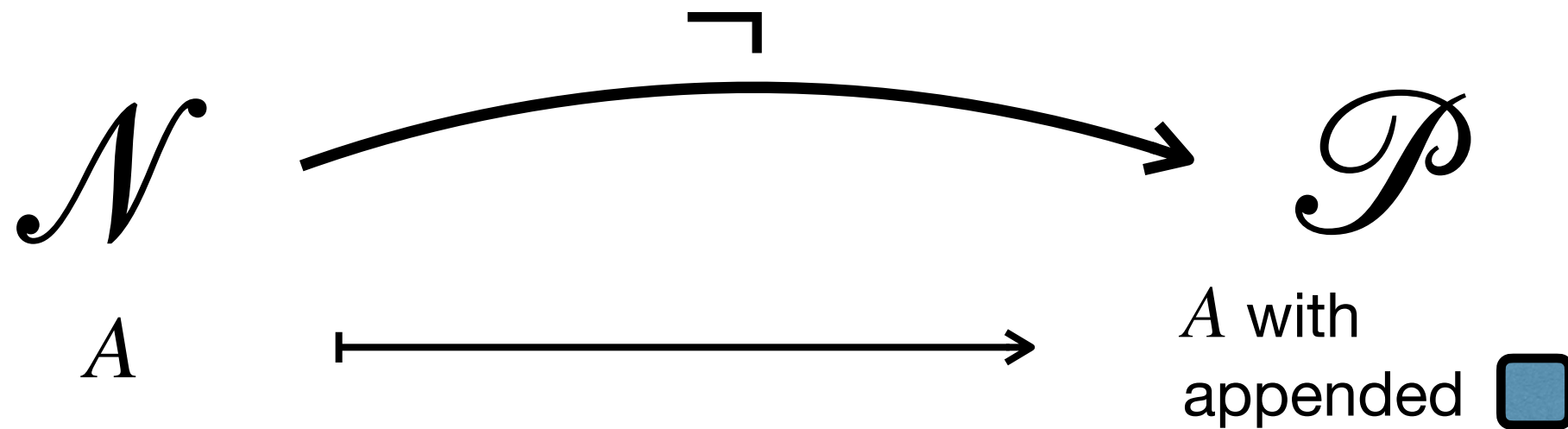
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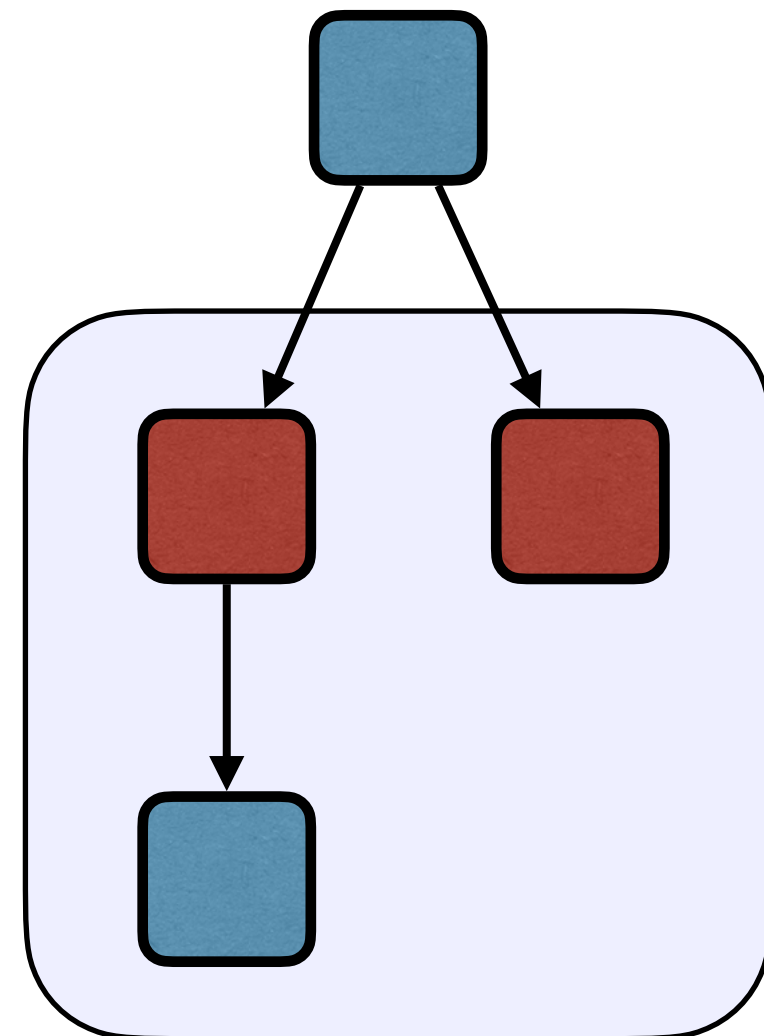
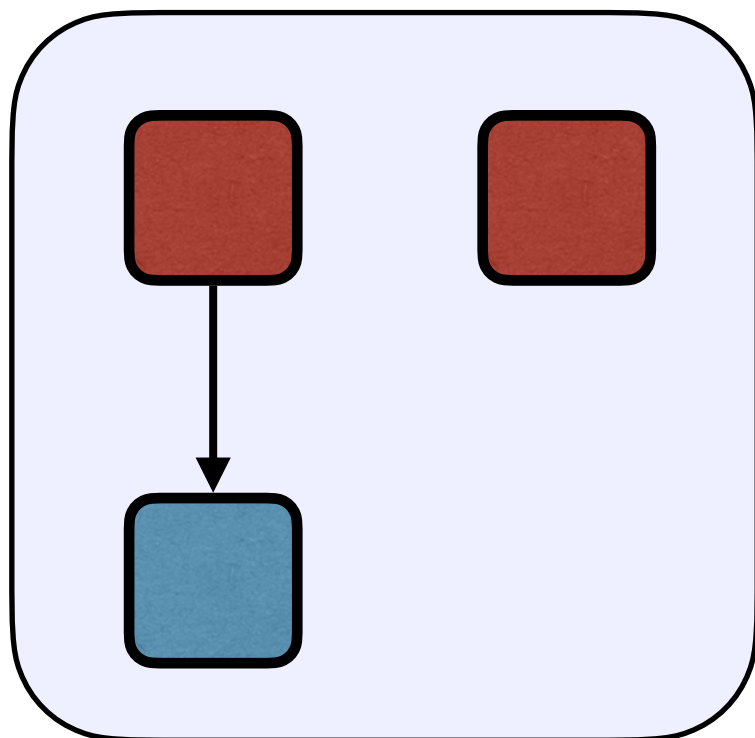
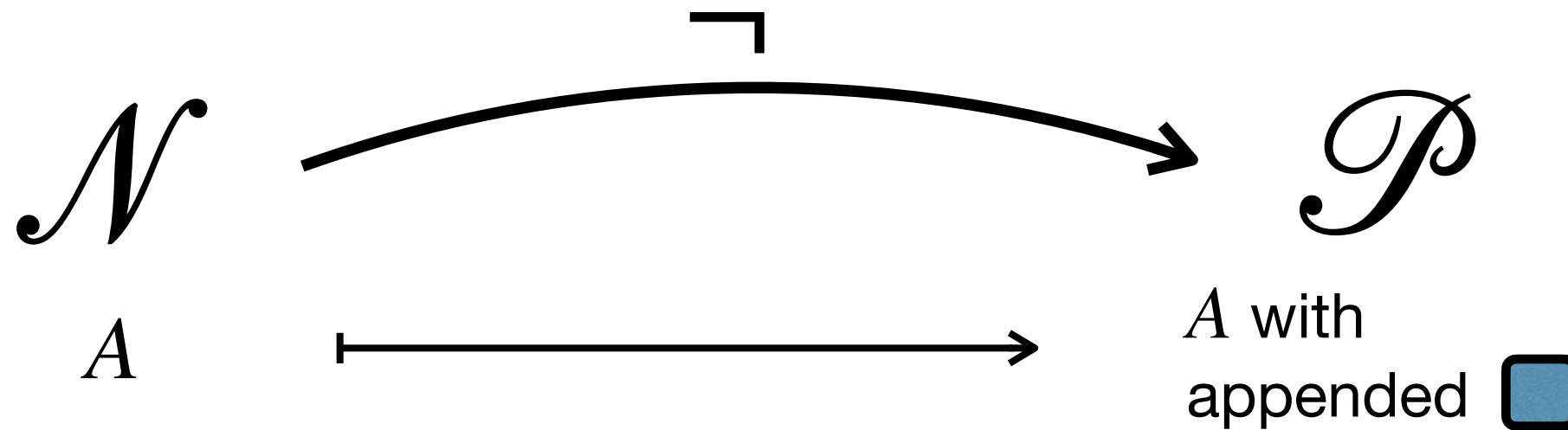
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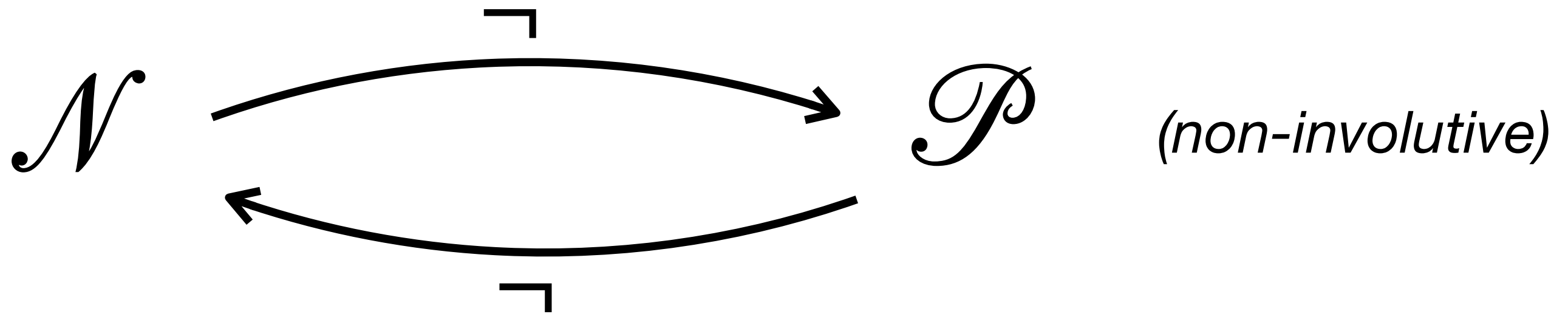
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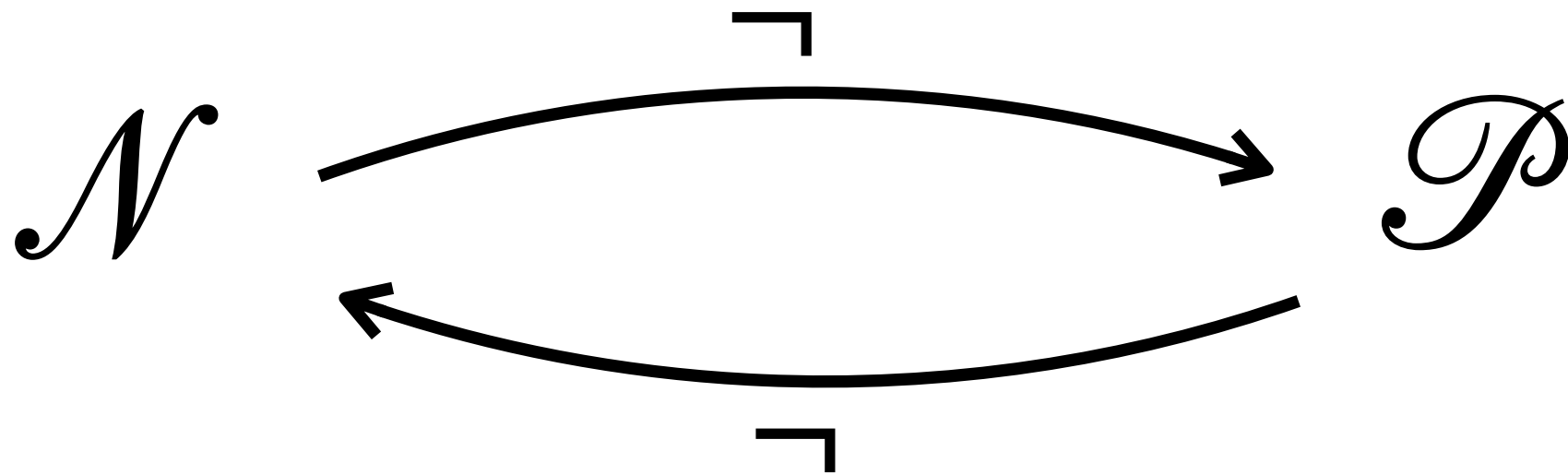
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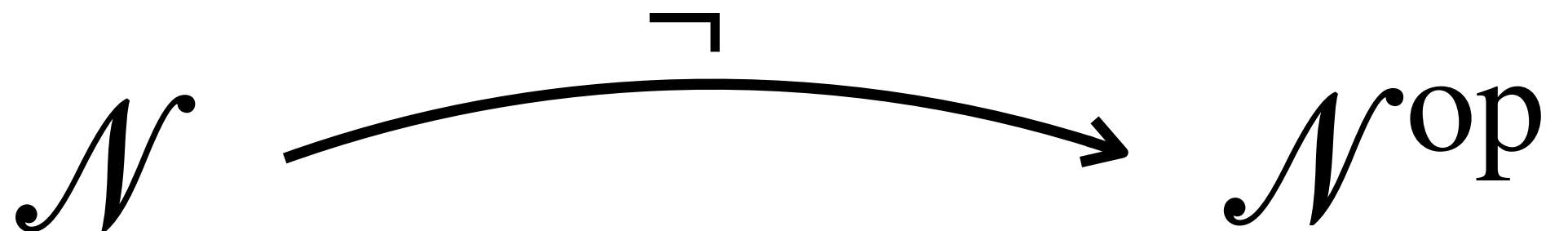


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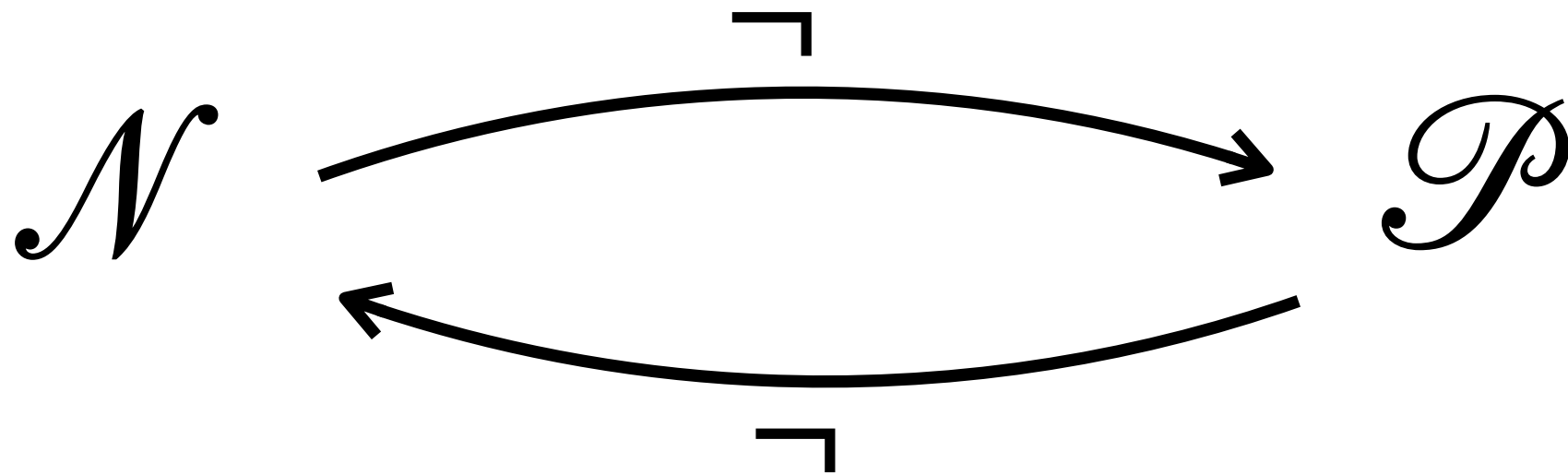


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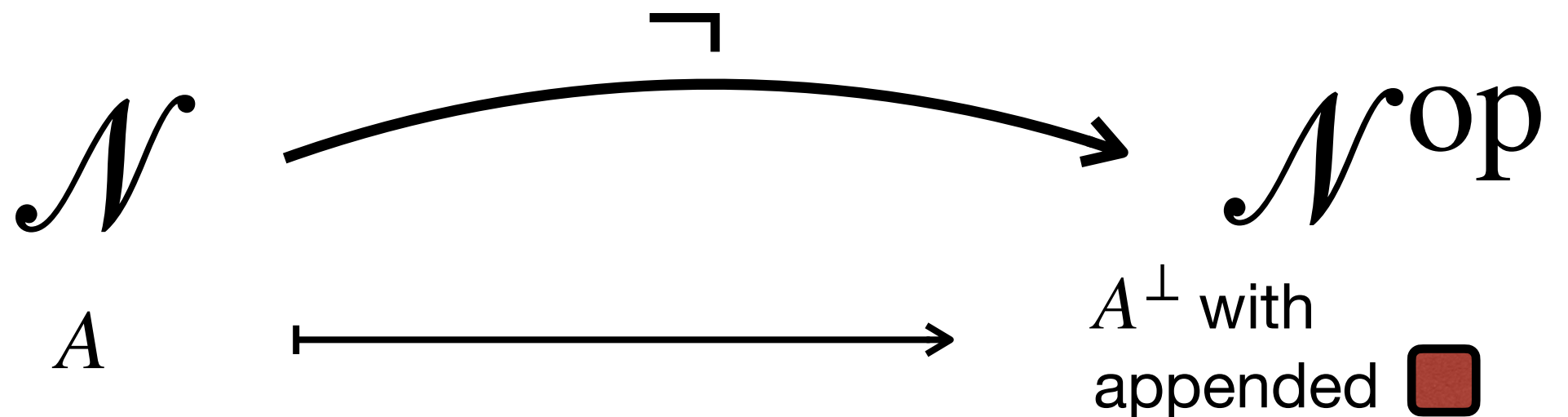


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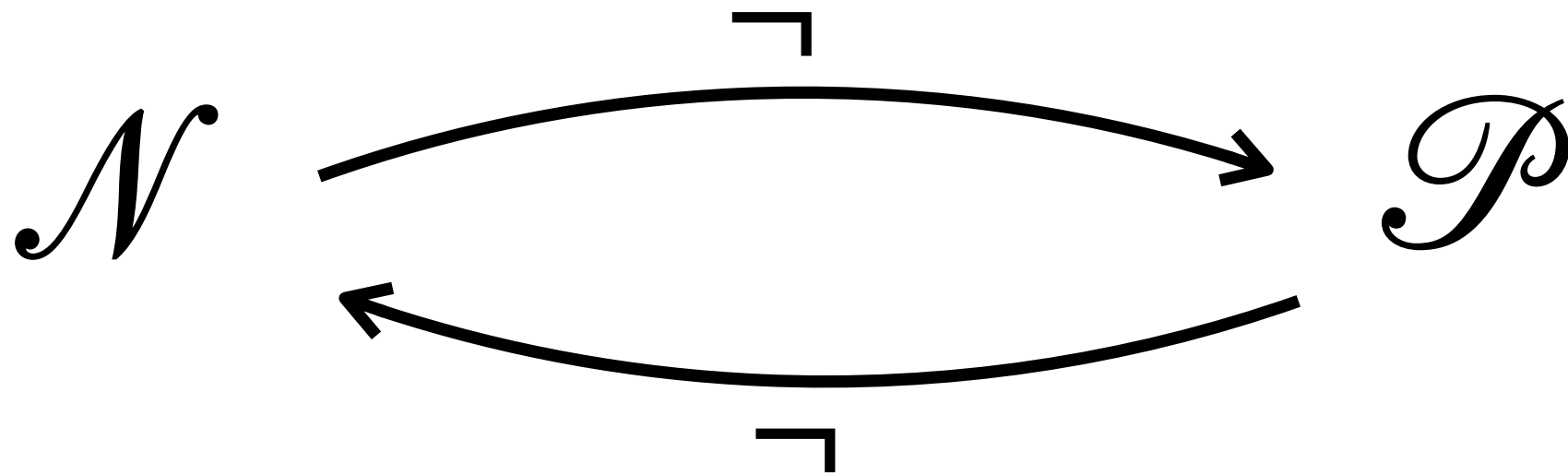


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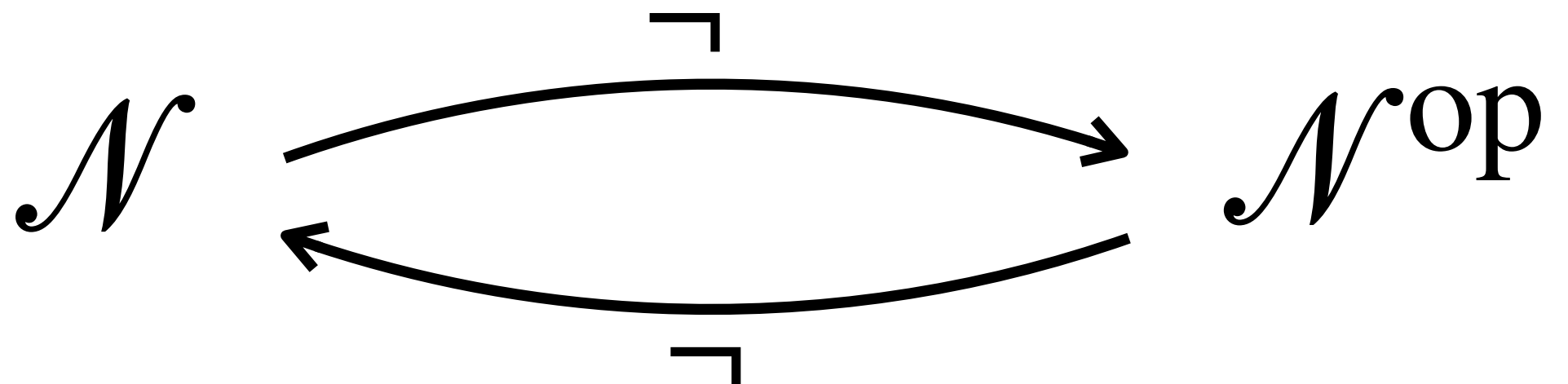


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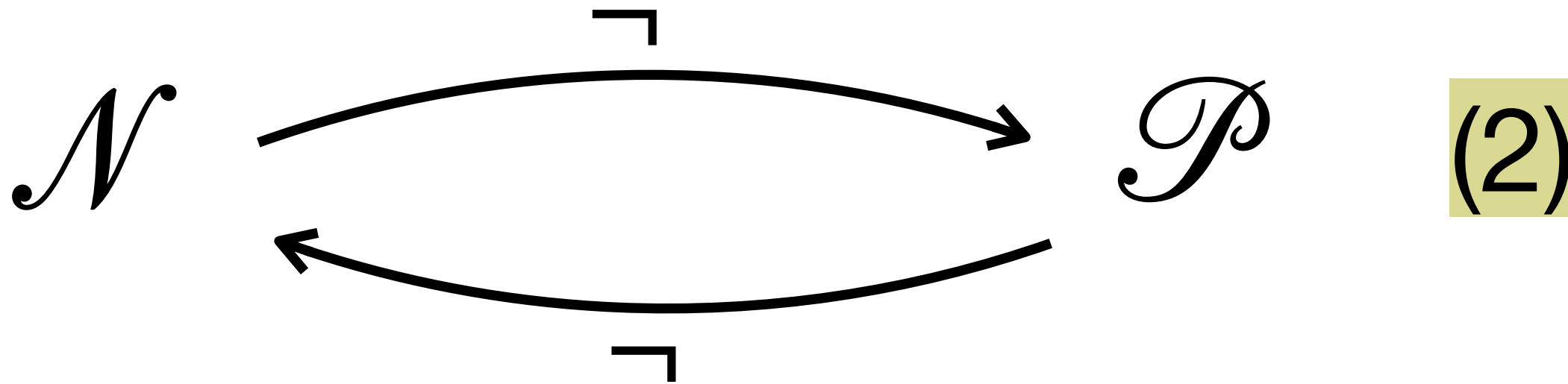


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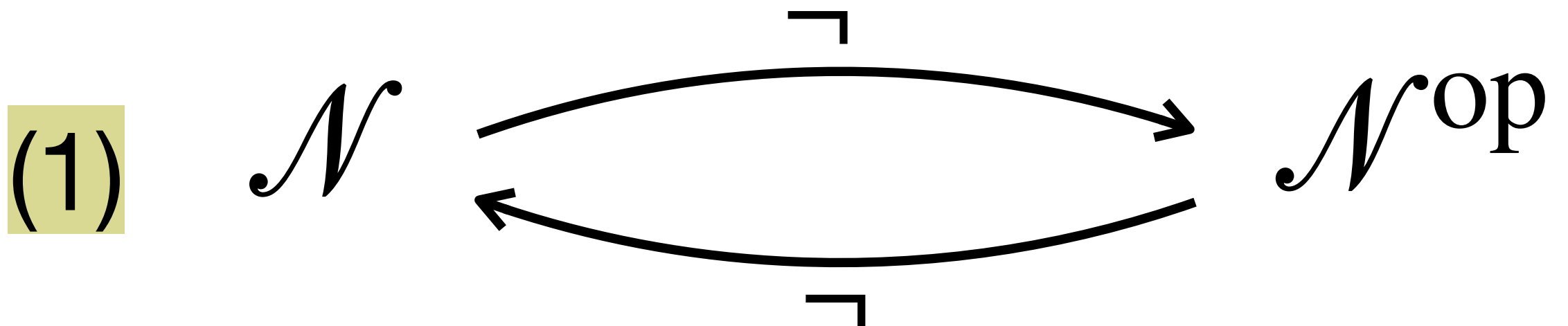


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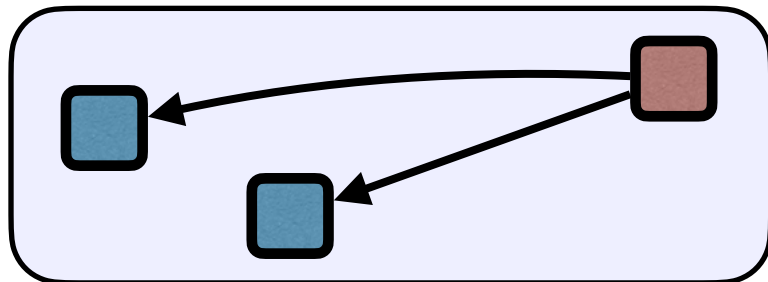


Concurrency

$$(1) \quad \mathcal{N} \begin{array}{c} \xrightarrow{\neg} \\ \xleftarrow{\neg} \end{array} \mathcal{N}^{\text{op}}$$

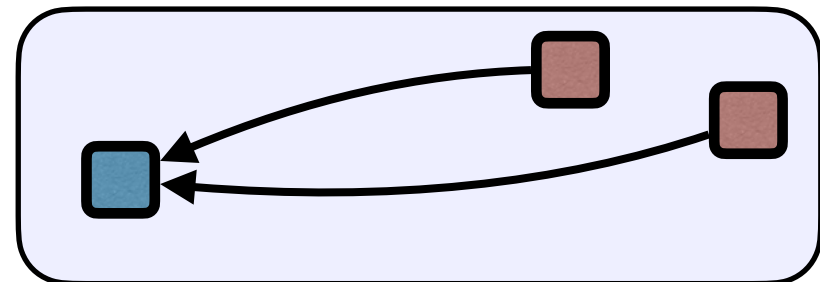
- Want combinators & axioms for concurrency
- **Key idea:** negation equipped with (semi) monoidal structure

$$\neg A \otimes \neg B \xrightarrow{\text{fork}_{A,B}} \neg(A \otimes B)$$



...

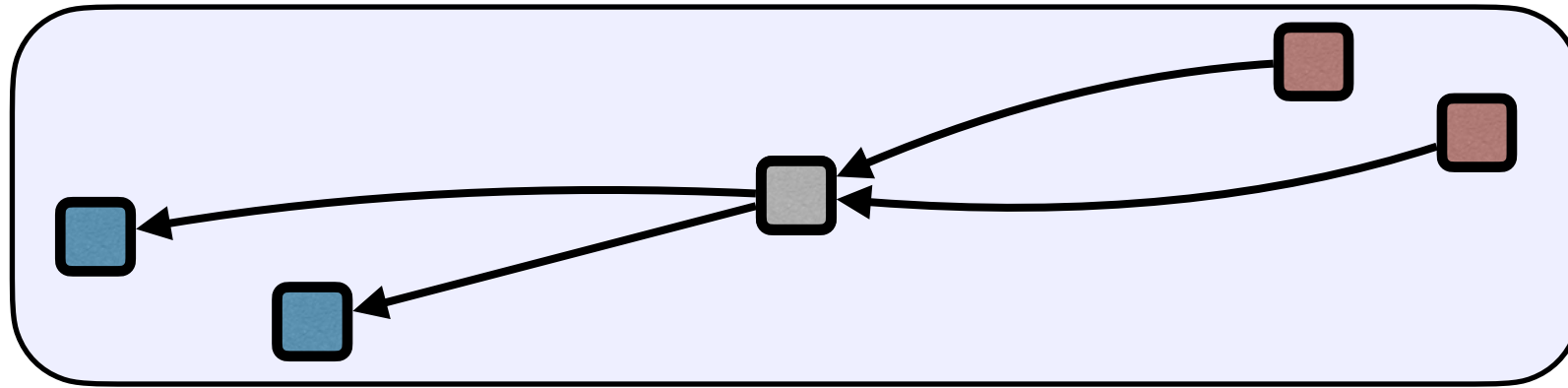
$$\neg(A \otimes B) \xrightarrow{\text{join}_{A,B}} \neg A \otimes \neg B$$



...

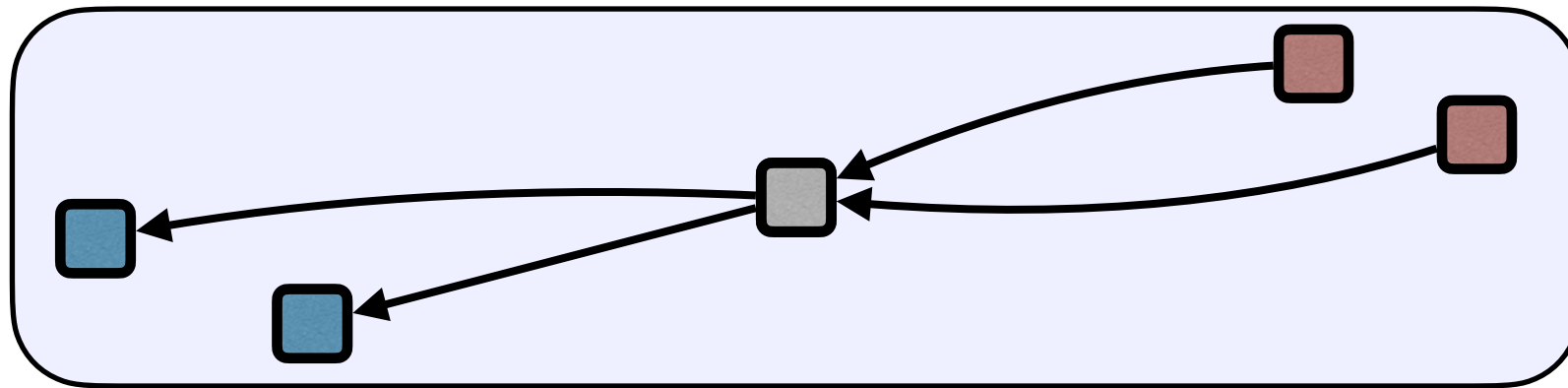
Concurrency (1) $\mathcal{N} \xrightarrow{\neg} \mathcal{N}^{\text{op}}$

$$\neg A \otimes \neg B \xrightarrow{\text{fork}_{A,B}} \neg(A \otimes B) \xrightarrow{\text{join}_{A,B}} \neg A \otimes \neg B$$



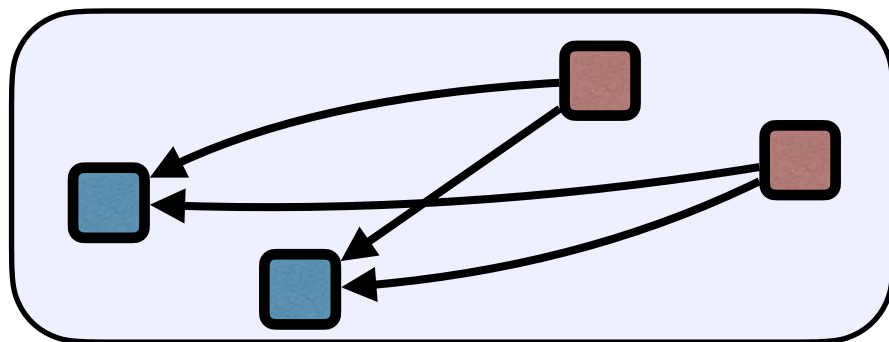
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$$\neg A \otimes \neg B \xrightarrow{\text{fork}_{A,B}} \neg(A \otimes B) \xrightarrow{\text{join}_{A,B}} \neg A \otimes \neg B$$



=

$$\neg A \otimes \neg B \xrightarrow{\text{join}_{A,B} \circ \text{fork}_{A,B}} \neg A \otimes \neg B$$

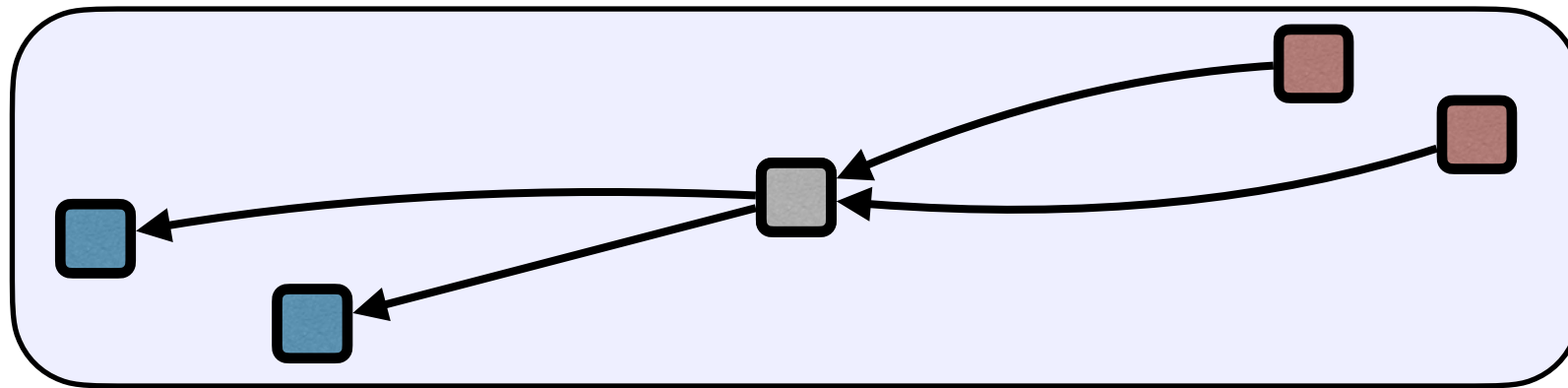


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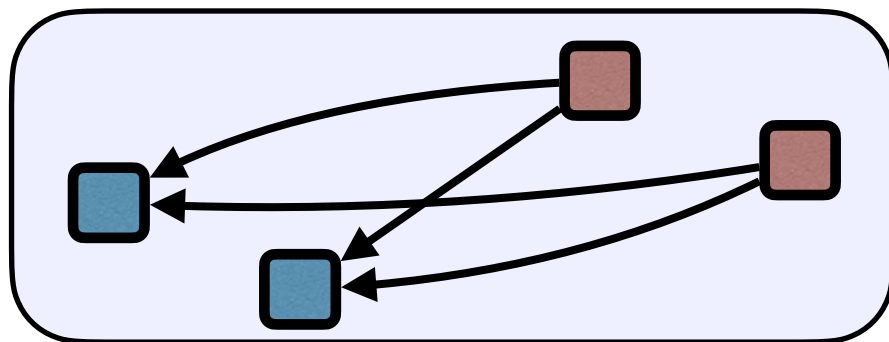
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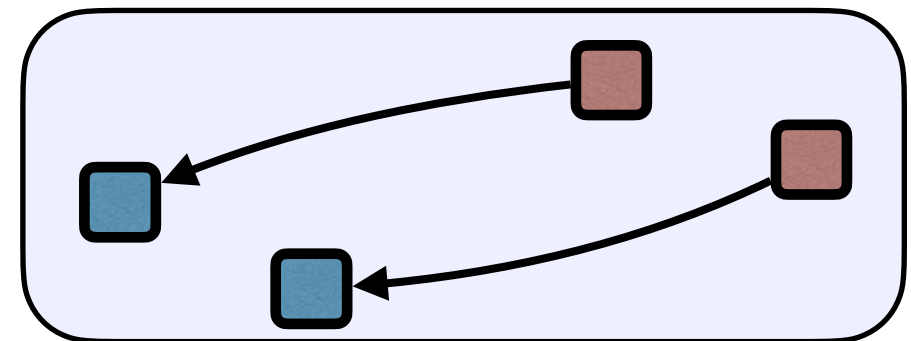


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$$\neg A \otimes \neg B \xrightarrow{\text{id}_{\neg A \otimes \neg B}} \neg A \otimes \neg B$$



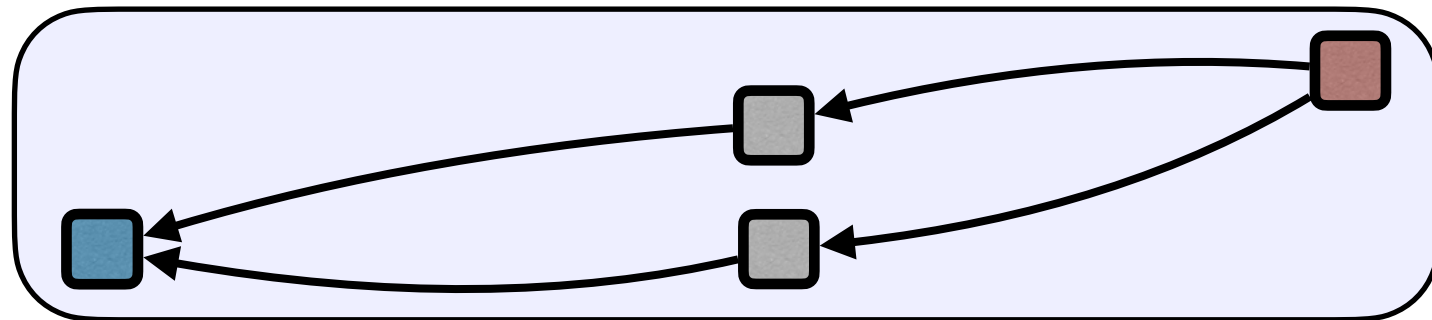
*weak
interchange
2-cell*

Concurrency

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$$\neg(A \otimes B) \xrightarrow{\text{join}_{A,B}} \neg A \otimes \neg B \xrightarrow{\text{fork}_{A,B}} \neg(A \otimes B)$$

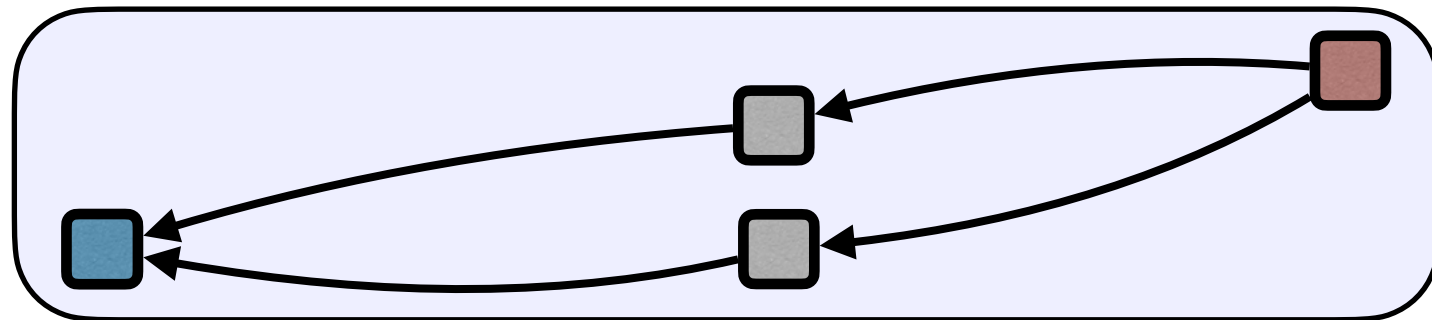


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(1)

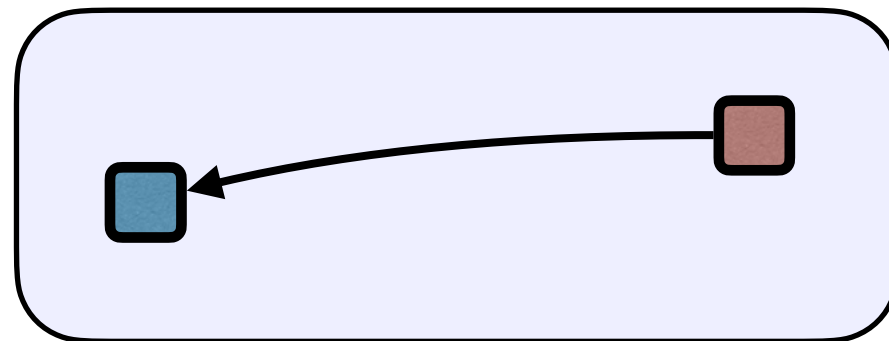
$$\mathcal{N} \begin{array}{c} \xrightarrow{\neg} \\ \xleftarrow{\neg} \end{array} \mathcal{N}^{\text{op}}$$

$$\neg(A \otimes B) \xrightarrow{\text{join}_{A,B}} \neg A \otimes \neg B \xrightarrow{\text{fork}_{A,B}} \neg(A \otimes B)$$



$$\neg(A \otimes B) \xrightarrow{\text{id}_{\neg(A \otimes B)}} \neg(A \otimes B)$$

=



Dialogue categories

Def. In a symmetric monoidal category $(\mathcal{C}, \otimes, I)$, a **return object** is an object $\perp$ equipped with a choice of function space $A \multimap \perp$ for every A .

The object $A \multimap \perp$ is denoted $\neg A$.

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Def. A **dialogue category** is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ equipped with a return object $\perp$.

Automatically: negation is a functor

$$\neg : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$$

with convenient properties. In particular $\neg \neg$ is a strong monad.

(1) $\mathcal{N} \begin{array}{c} \xrightarrow{\neg} \\ \perp \\ \xleftarrow{\neg} \end{array} \mathcal{N}^{\text{op}}$

Dialogue 2-categories

(Needed because the weak interchange map is a 2-cell.)

Def. A **dialogue 2-category** is a symmetric monoidal 2-category $(\mathcal{C}, \otimes, I)$ equipped with a return object $\perp$ (where the function space $A \multimap \perp$ has a 2-categorical universal property).

Many examples:

- Game semantics
- Models of LL.
- Continuation models: response categories, continuation categories.

(some are *bicategories*)

Concurrent dialogue 2-categories

Definition. A concurrent dialogue 2-category is a dialogue 2-category equipped with:

- $\text{fork}_{A,B} : \neg A \otimes \neg B \rightarrow \neg(A \otimes B)$
- $\text{join}_{A,B} : \neg(A \otimes B) \rightarrow \neg A \otimes \neg B$
- $\sigma_{A,B} :$

$$\begin{array}{ccc} & \neg(A \otimes B) & \\ \text{fork}_{A,B} \nearrow & \Downarrow \sigma_{A,B} & \searrow \text{join}_{A,B} \\ \neg A \otimes \neg B & \xlongequal{\quad} & \neg A \otimes \neg B \end{array}$$

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 \end{array}$$

Axioms:

$$\begin{array}{ccc}
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- naturality and modification
- monoidal associativity, units

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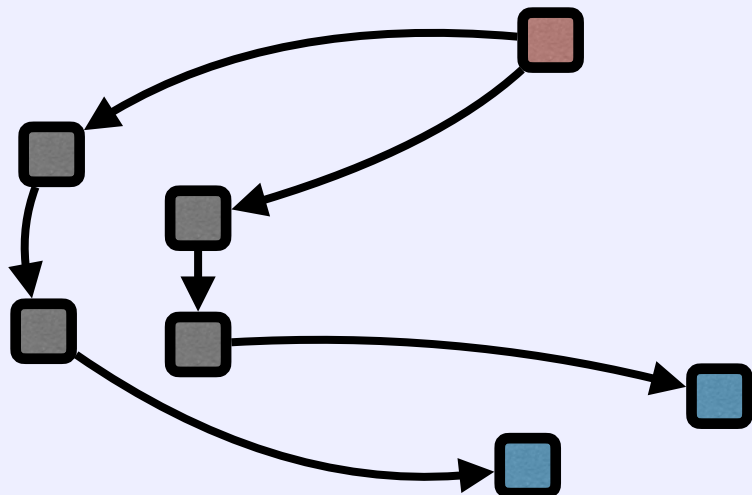
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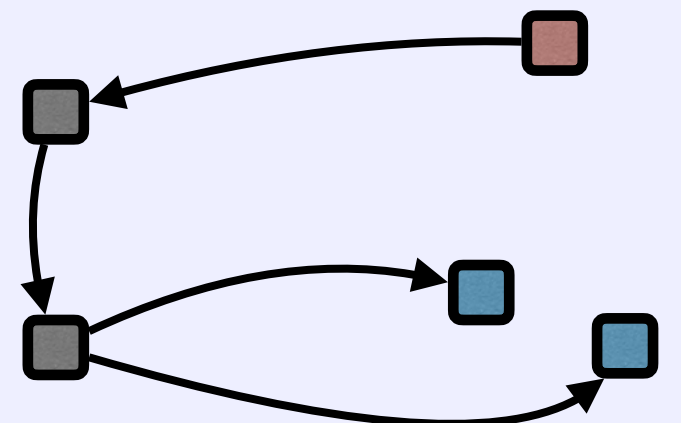
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$$\neg \neg A \otimes \neg \neg B \xrightarrow{\text{fork}_{\neg A, \neg B}} \neg(\neg A \otimes \neg B)$$



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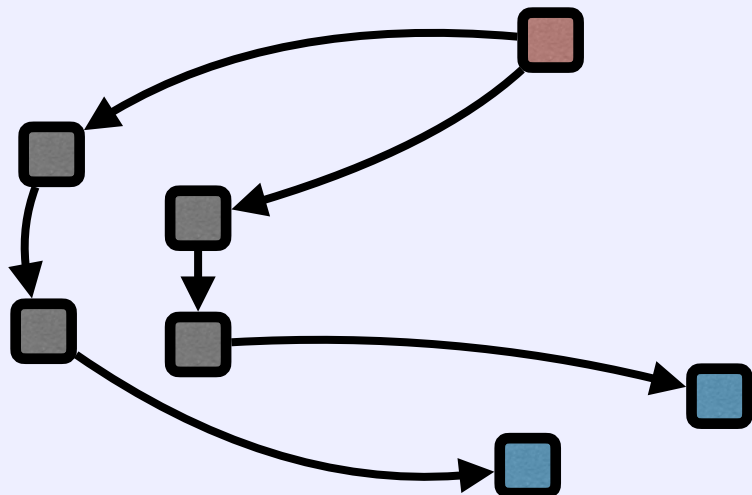
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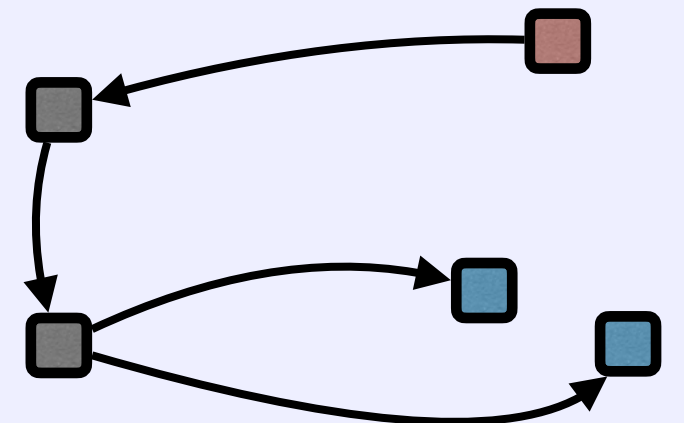
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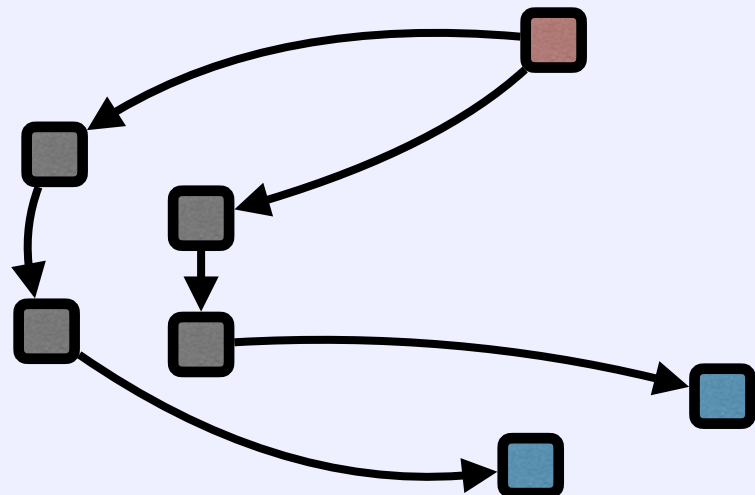
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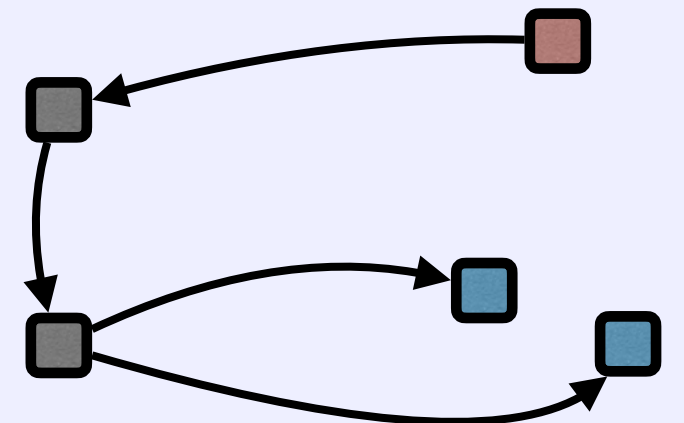
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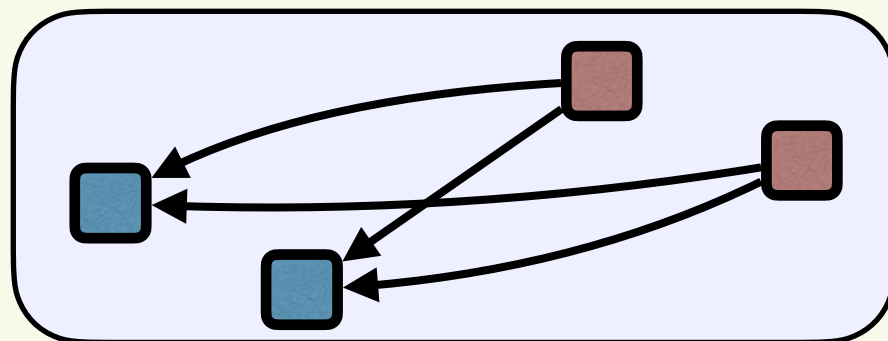
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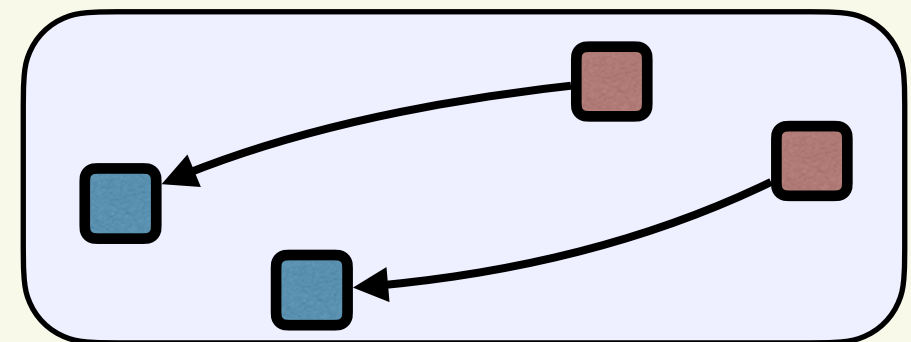
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$$\xRightarrow{\sigma_{A,B}}$$



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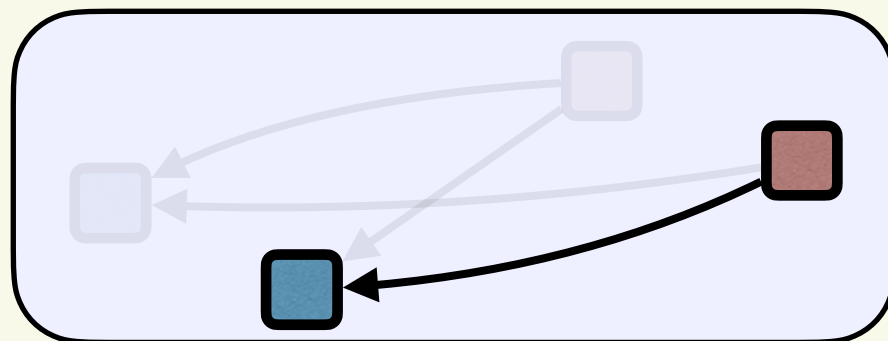
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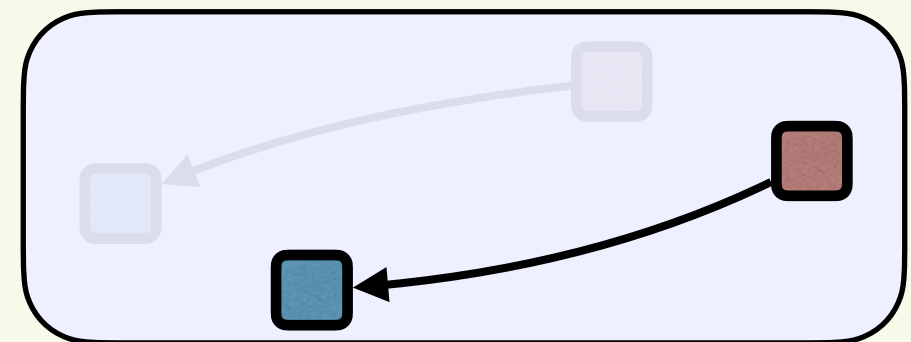
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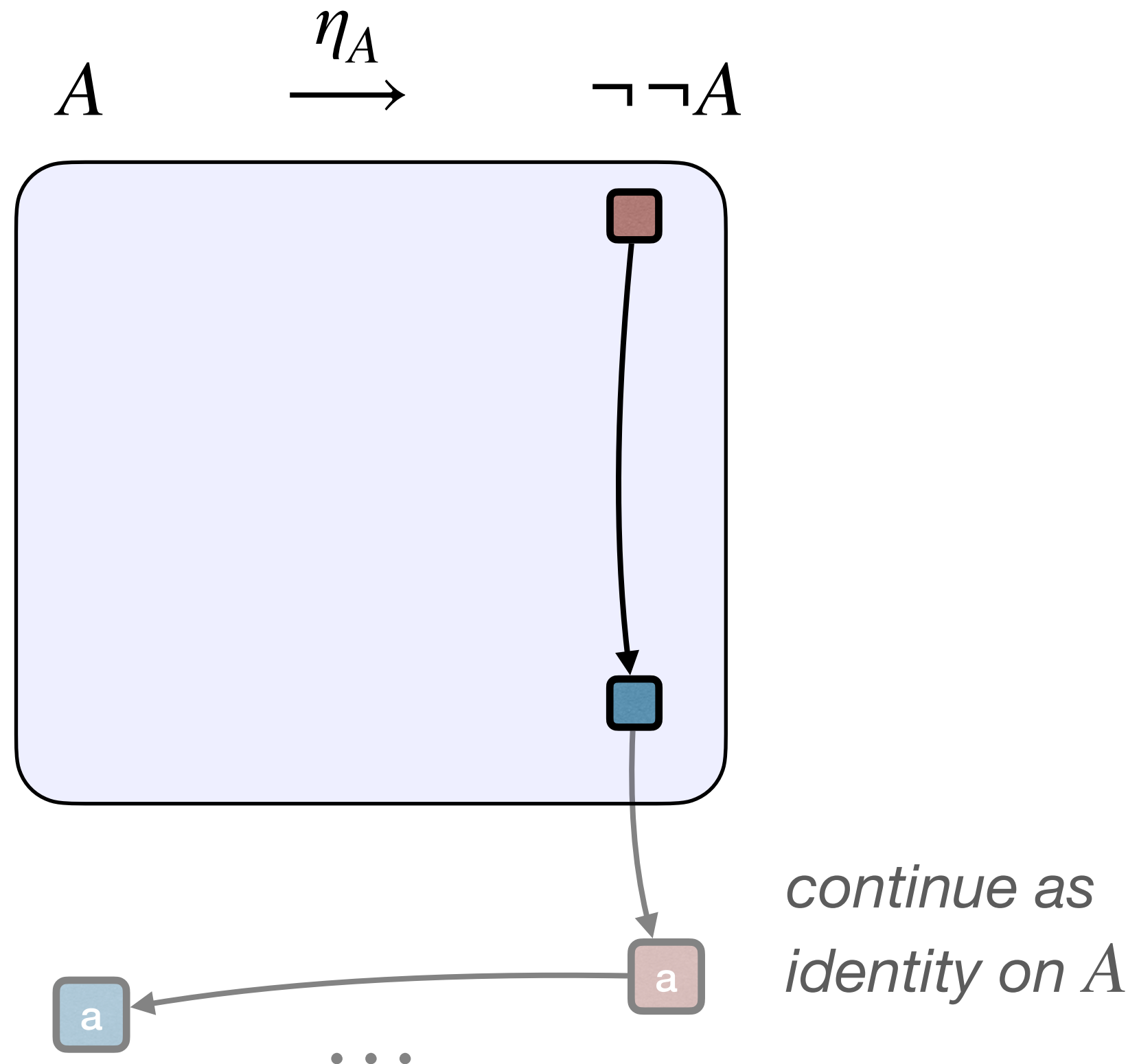
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 \end{array}$$

- naturality and modification
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Theorem: in a concurrent dialogue 2-category, $\neg \neg$ is a concurrent 2-monad.

The double negation 2-monad

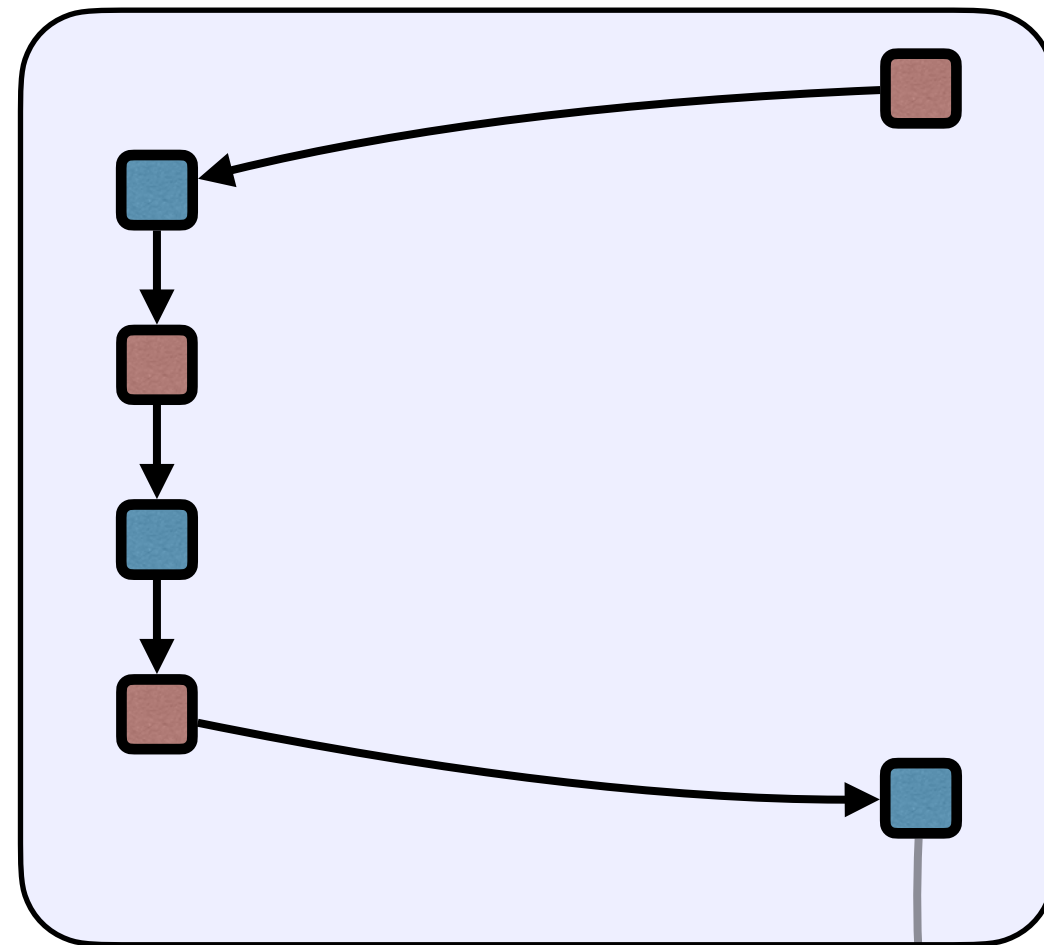


The double negation 2-monad

$$\neg\neg\neg\neg A \xrightarrow{\mu_A} \neg\neg A$$

The double negation 2-monad

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*continue as
identity on A*

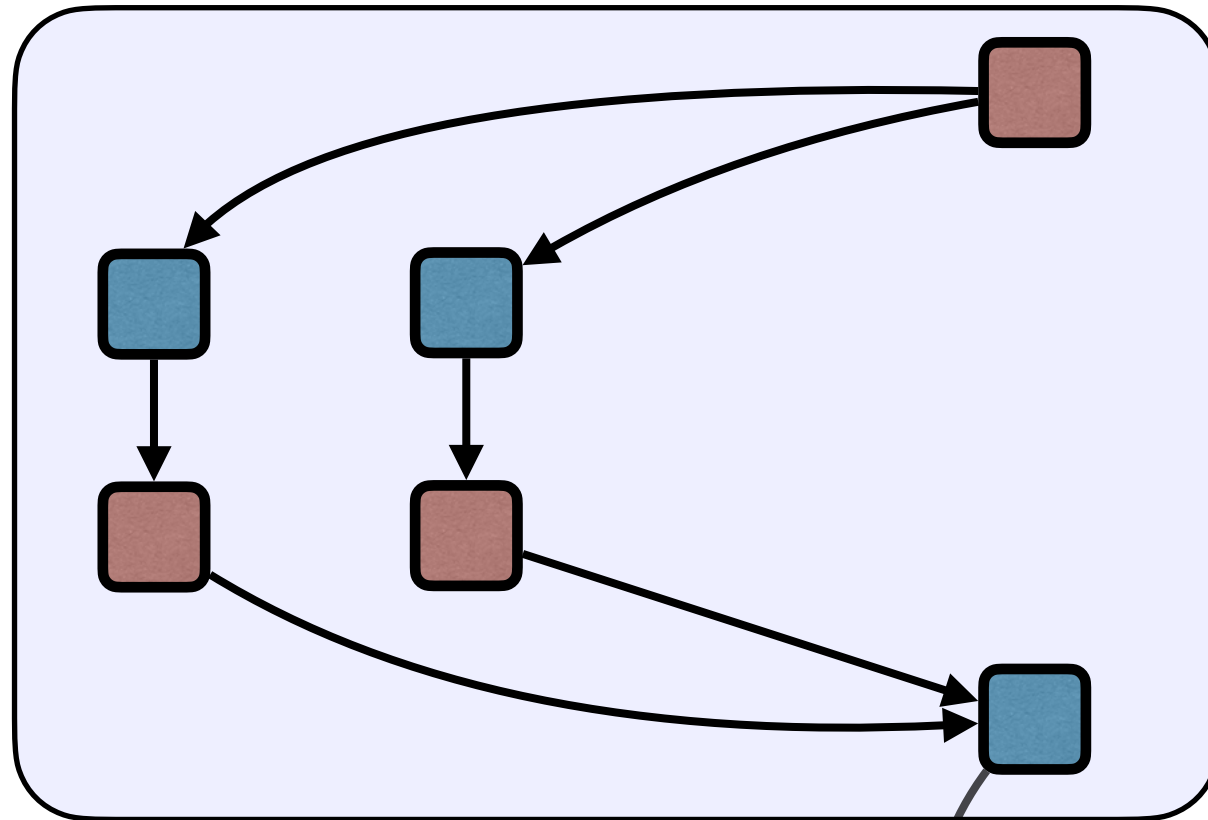


...

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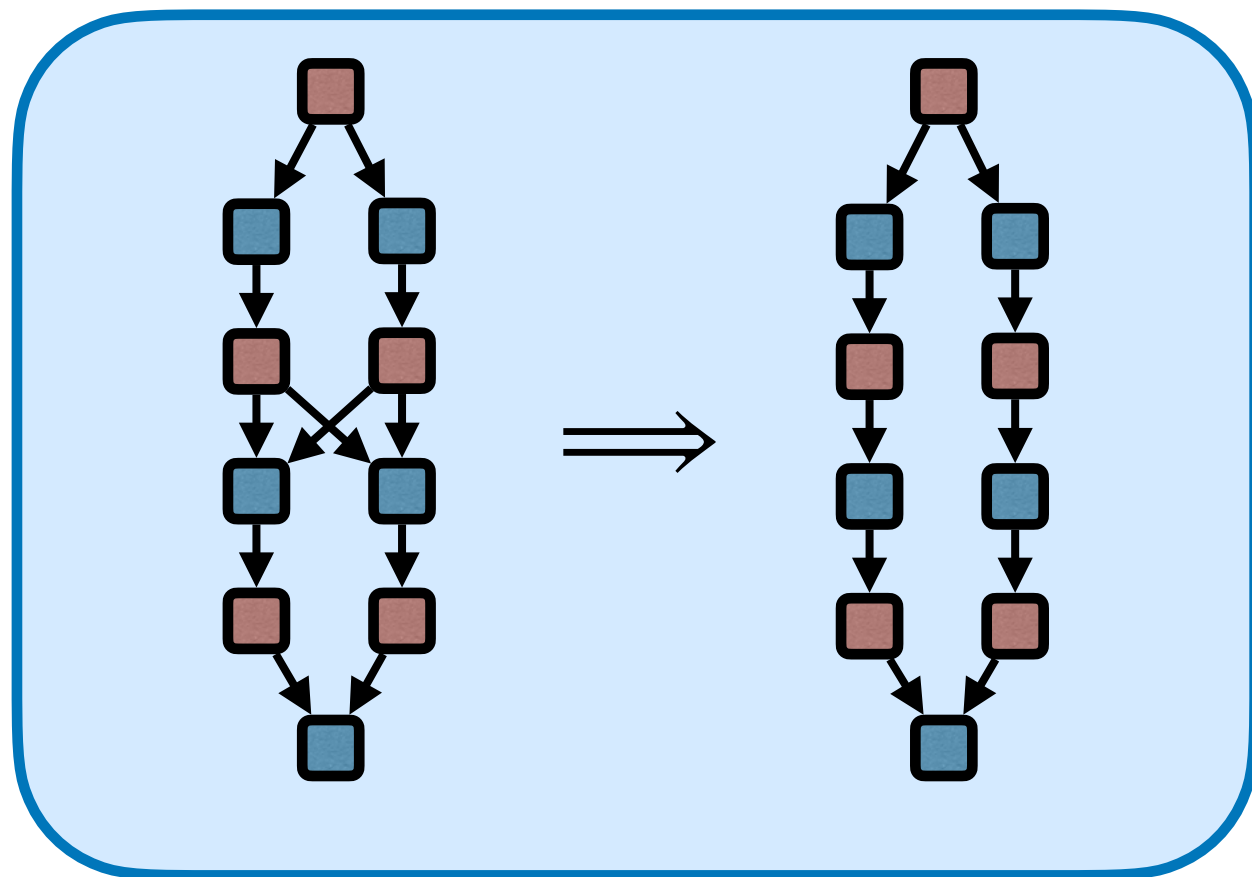
$$\neg\neg A \quad \otimes \quad \neg\neg B \quad \xrightarrow{\chi_{A,B}} \quad \neg\neg(A \otimes B)$$



*continue as
identity on $A \otimes B$*

...

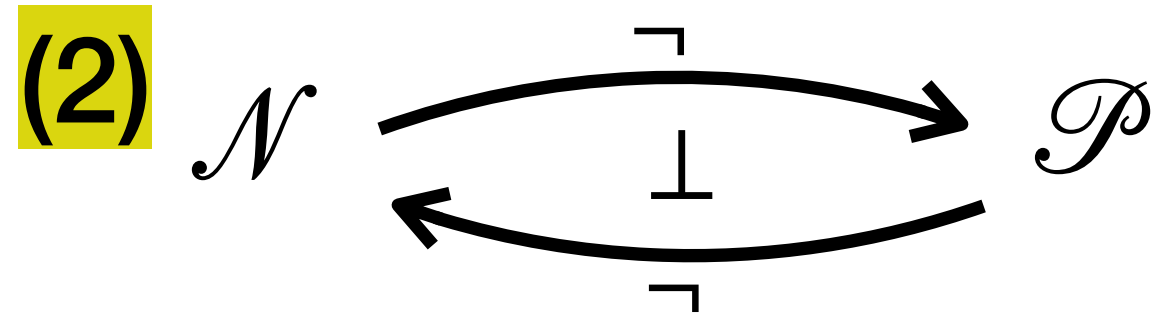
$$\begin{array}{ccccc}
TTA \otimes TTB & \xrightarrow{\chi} & T(TA \otimes TB) & \xrightarrow{T\chi} & TT(A \otimes B) \\
\mu \otimes \mu \downarrow & & \swarrow & & \downarrow \mu \\
TA \otimes TB & \xrightarrow{\chi} & & & T(A \otimes B)
\end{array}$$



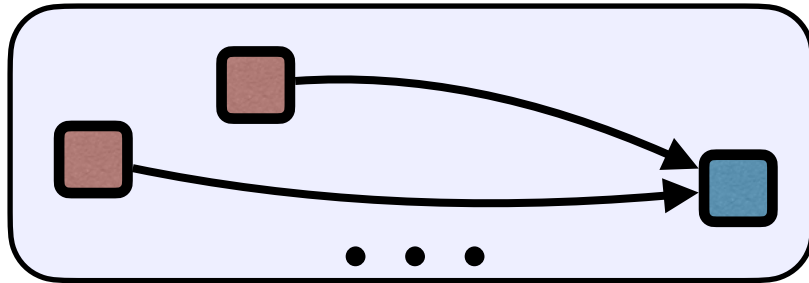
$$\begin{array}{ccccc}
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 \mu \otimes \mu \downarrow & & & \swarrow & \downarrow \mu \\
 TA \otimes TB & \xrightarrow{\chi} & & & T(A \otimes B)
 \end{array}$$

Concurrency

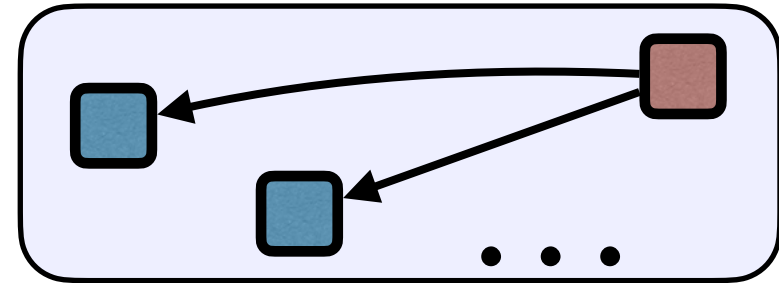
(2)



$$\neg N \otimes \neg M \xrightarrow{\text{join}_{N,M}} \neg(N \otimes M)$$

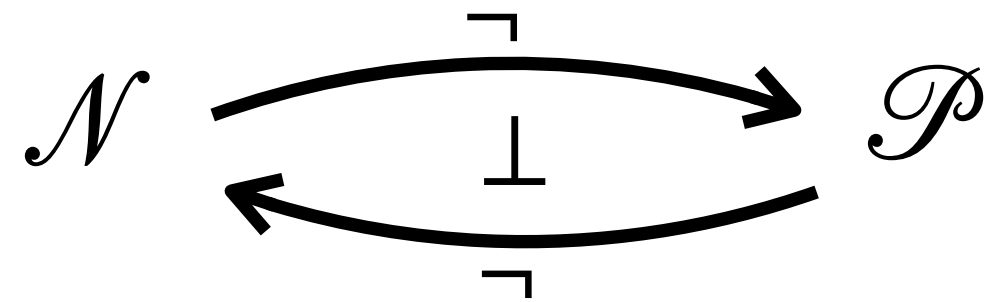


$$\neg P \otimes \neg Q \xrightarrow{\text{fork}_{P,Q}} \neg(P \otimes Q)$$

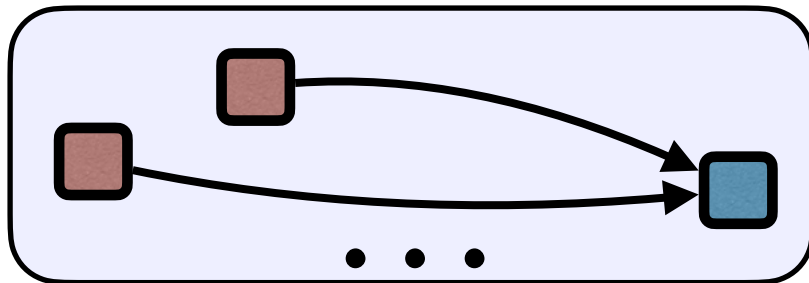


Concurrency

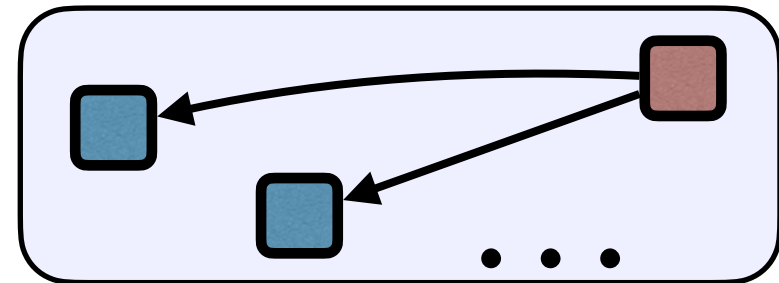
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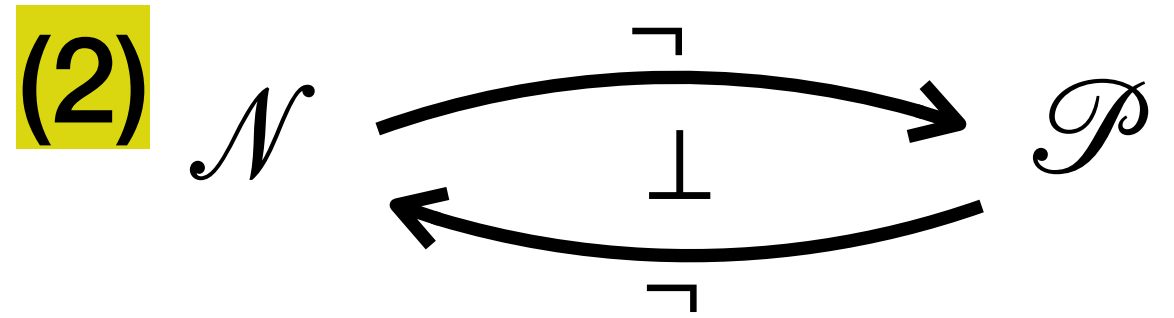


$$\neg P \otimes \neg Q \xrightarrow{\text{fork}_{P,Q}} \neg(P \otimes Q)$$

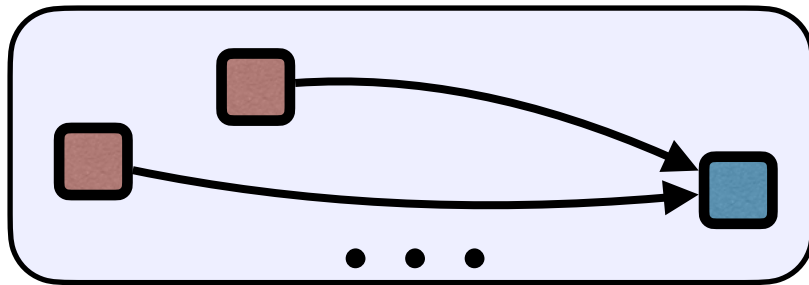


Can't directly compose them. But consider:

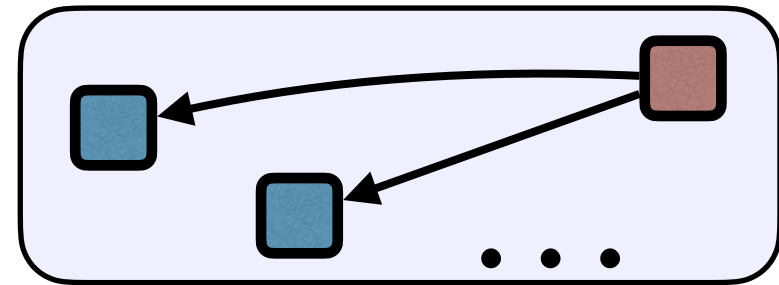
Concurrency



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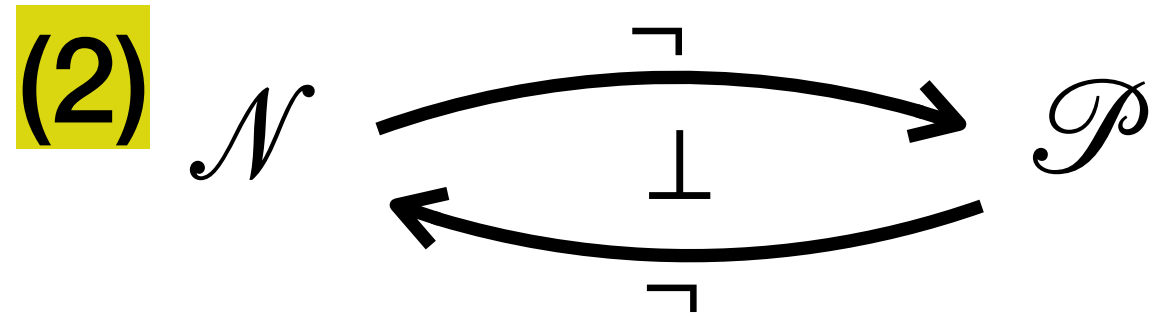
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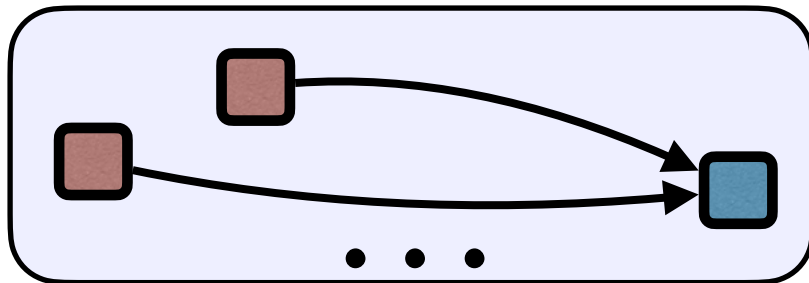
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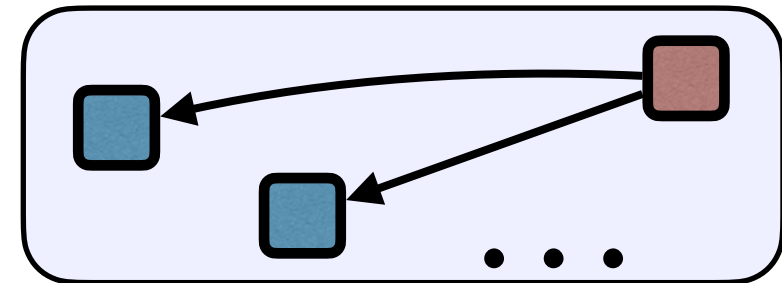
Concurrency



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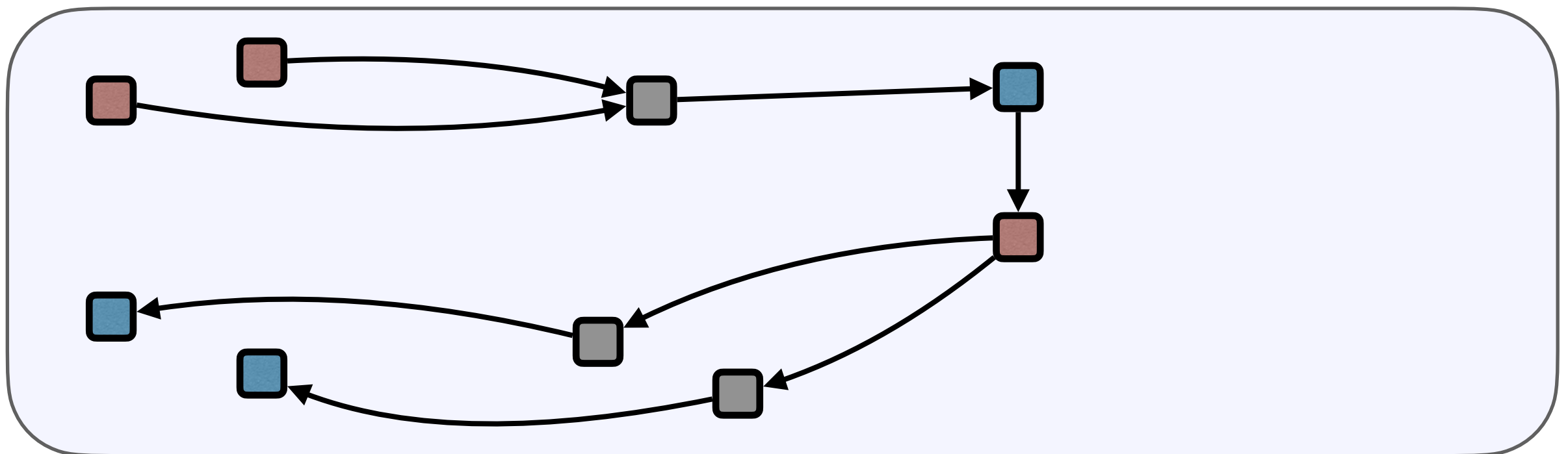


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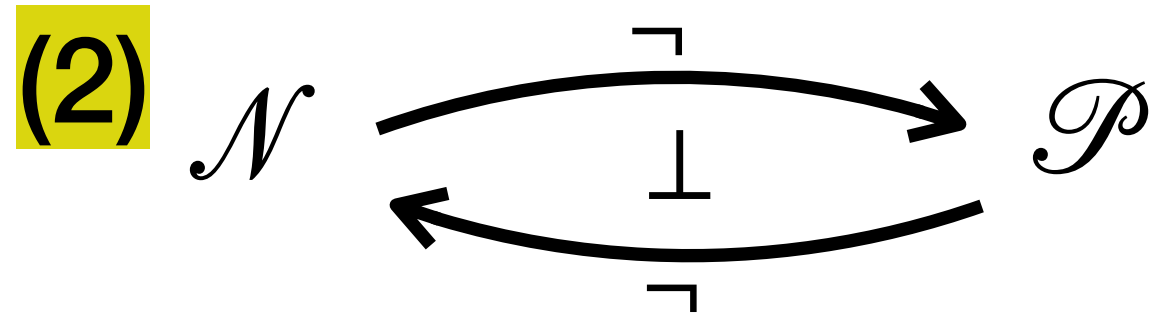


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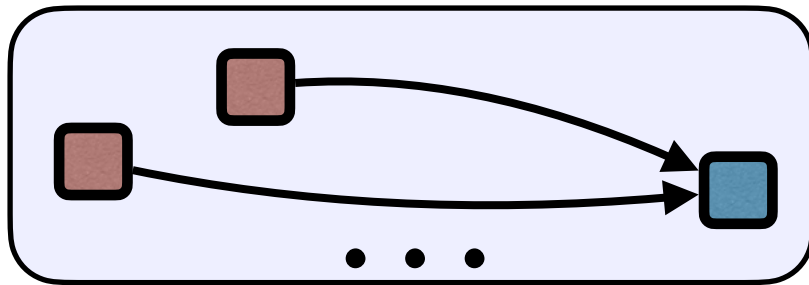
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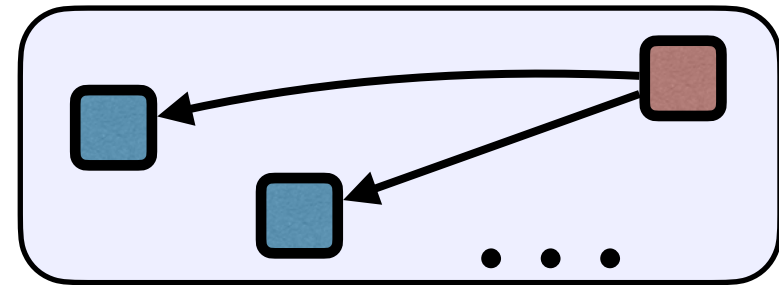
Concurrency



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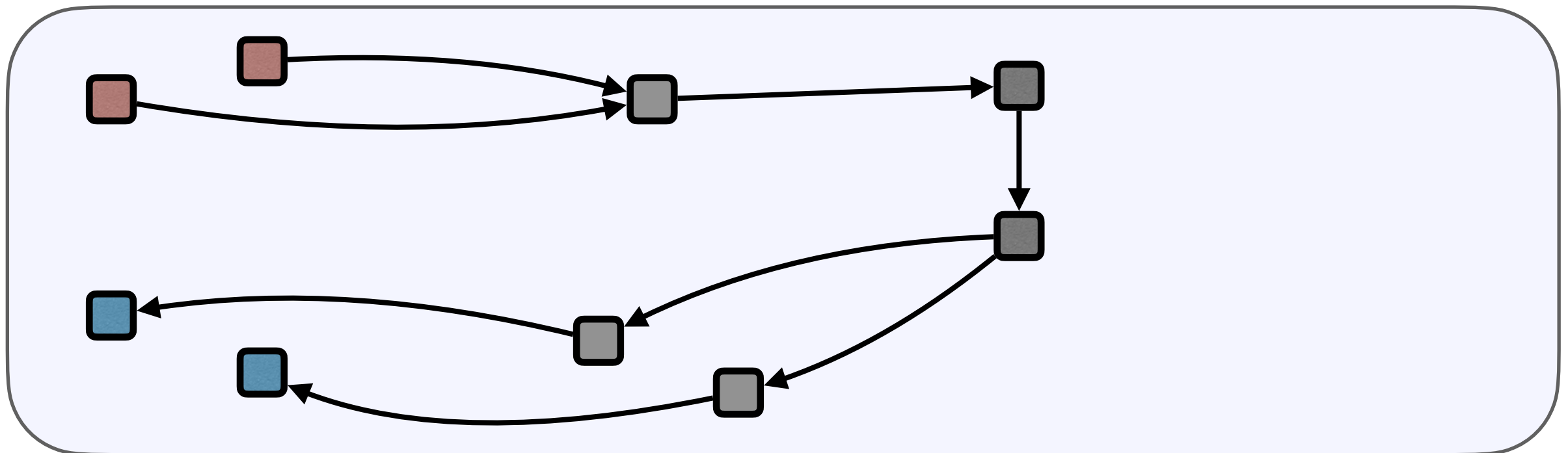


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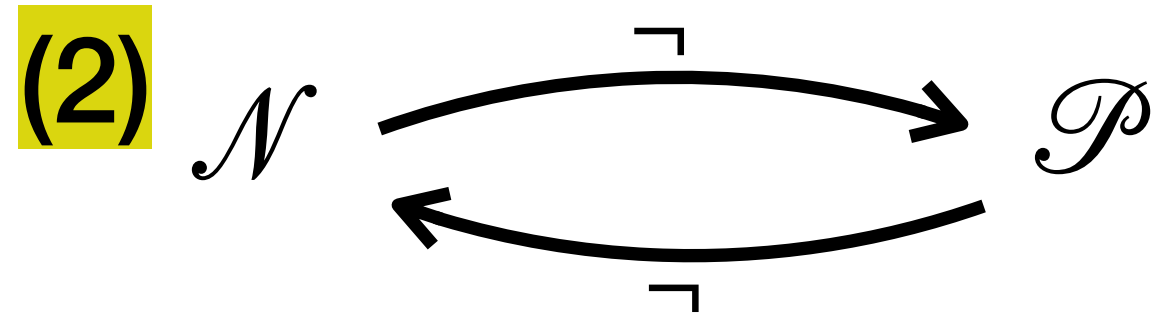
Can't directly compose them. But consider:

$$\neg\neg P \otimes \neg\neg Q \xrightarrow{\text{join}_{\neg P, \neg Q}} \neg(\neg P \otimes \neg Q) \xrightarrow{\neg\text{fork}_{P,Q}} \neg\neg(P \otimes Q) \xrightarrow{\epsilon} P \otimes Q$$

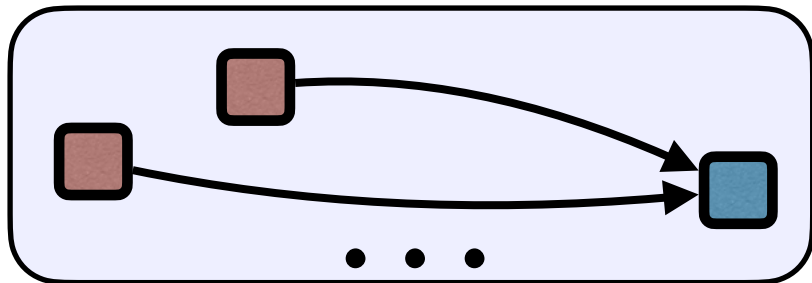


Concurrency

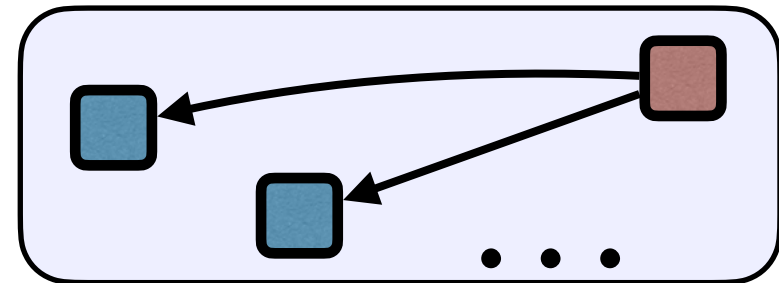
(2)



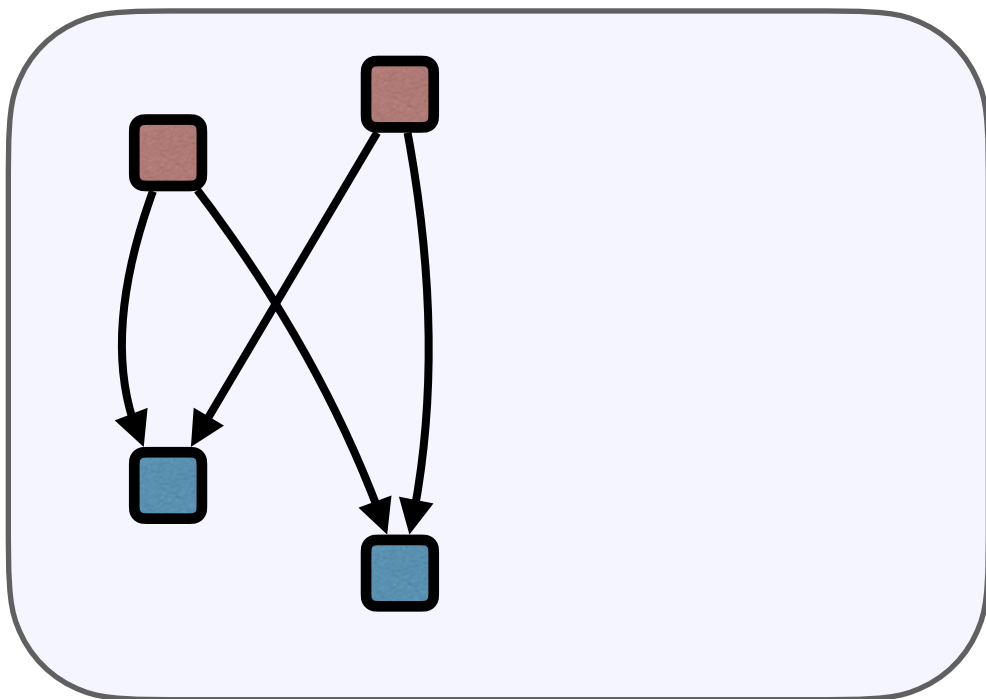
$$\neg N \otimes \neg M \xrightarrow{\text{join}_{N,M}} \neg(N \otimes M)$$



$$\neg P \otimes \neg Q \xrightarrow{\text{fork}_{P,Q}} \neg(P \otimes Q)$$

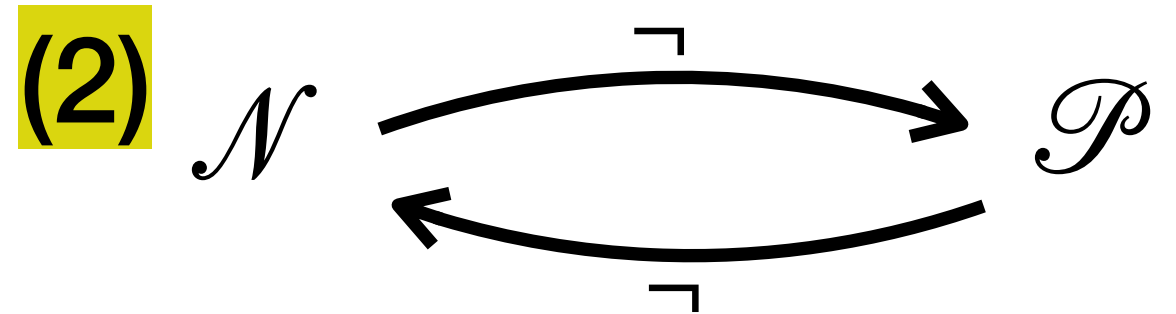


$$\neg\neg P \otimes \neg\neg Q \longrightarrow P \otimes Q$$

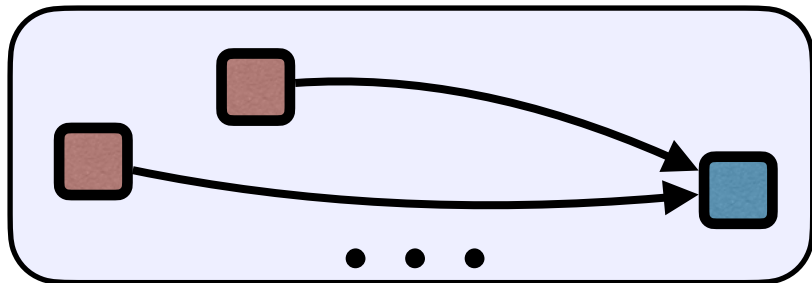


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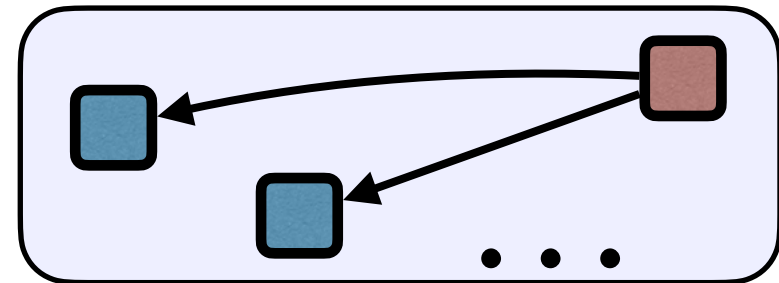
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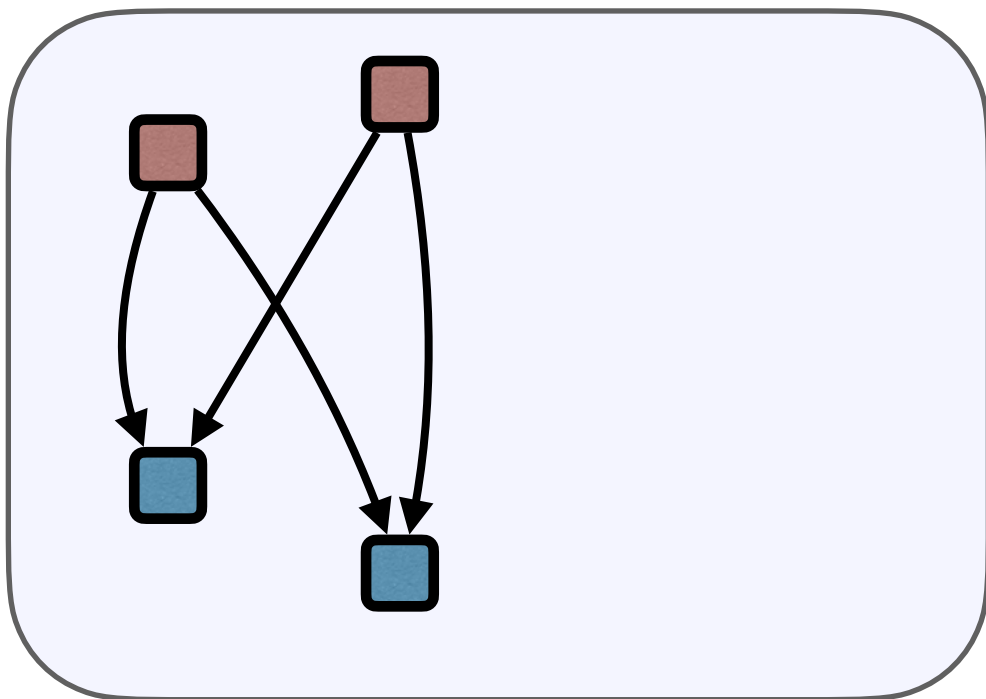
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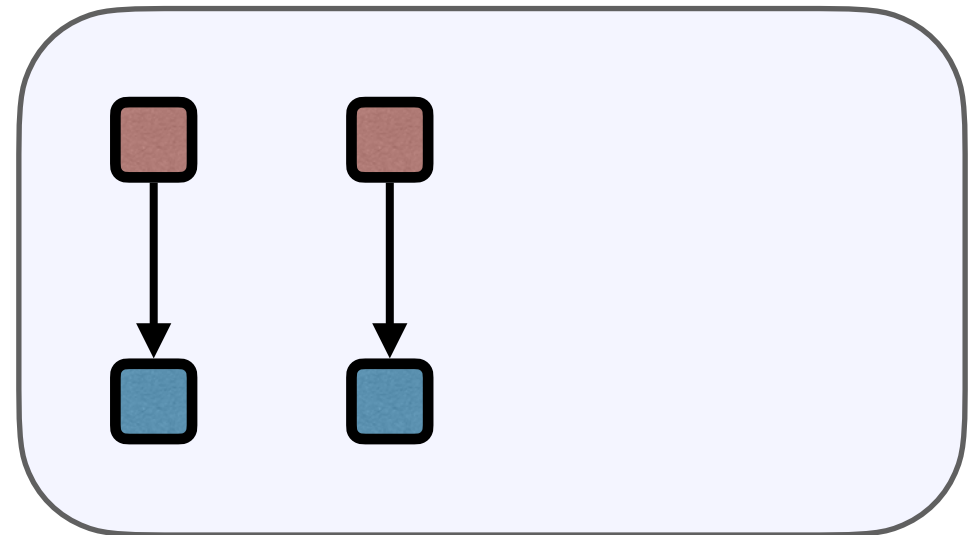
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$$\neg\neg P \otimes \neg\neg Q \longrightarrow P \otimes Q$$



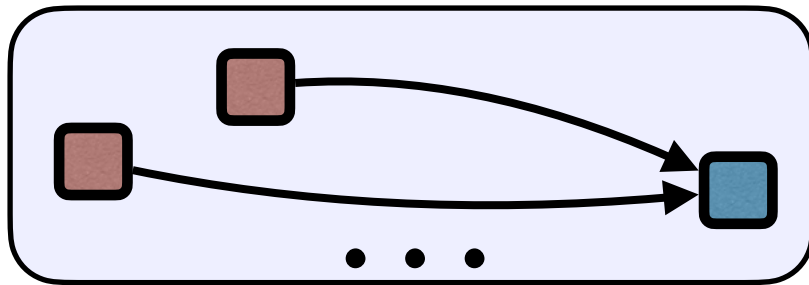
$$\neg\neg P \otimes \neg\neg Q \xrightarrow{\varepsilon \otimes \varepsilon} P \otimes Q$$



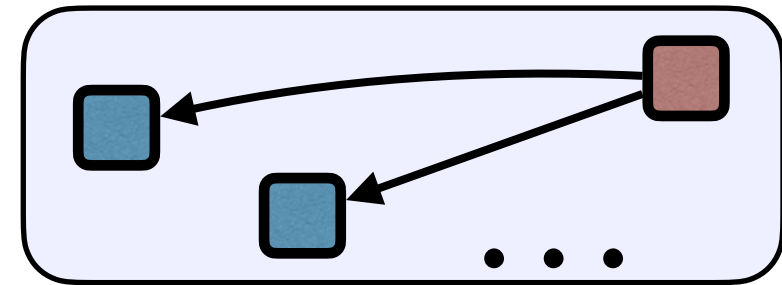
Concurrency

(2) $\mathcal{N} \xrightarrow{\neg} \mathcal{P}$

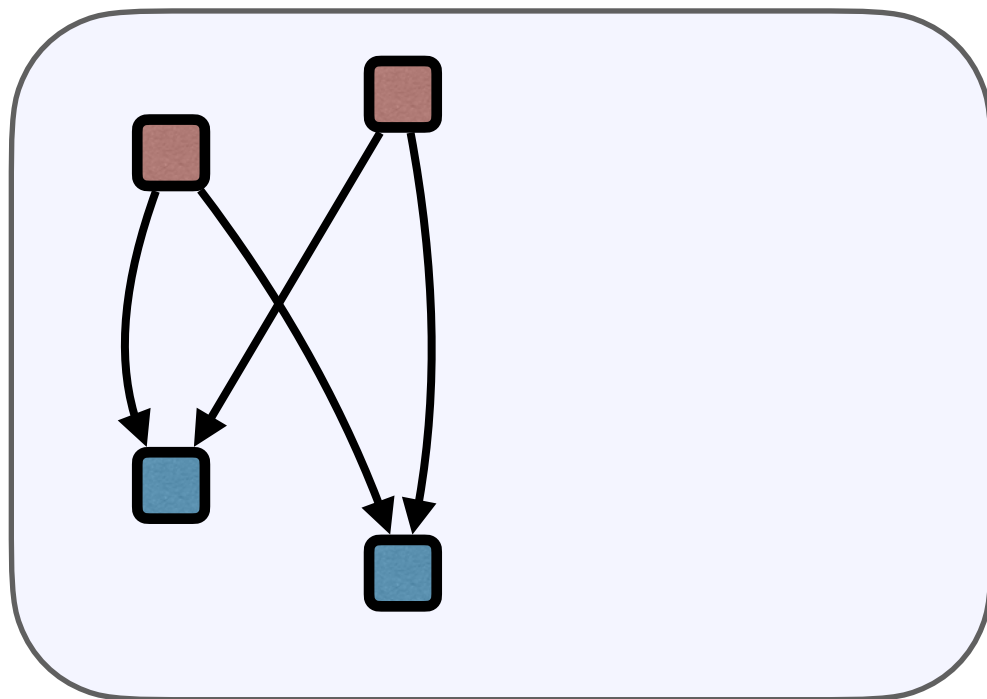
$$\neg N \otimes \neg M \xrightarrow{\text{join}_{N,M}} \neg(N \otimes M)$$



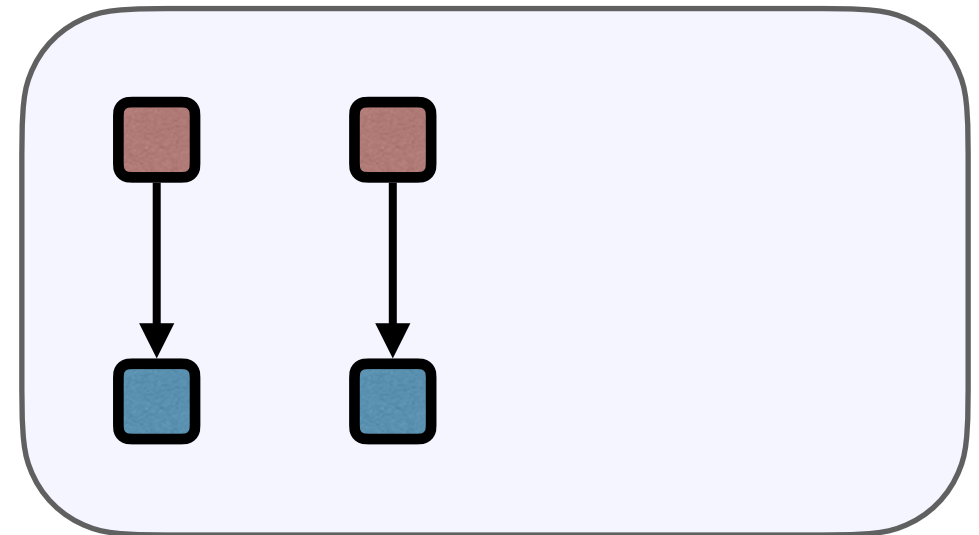
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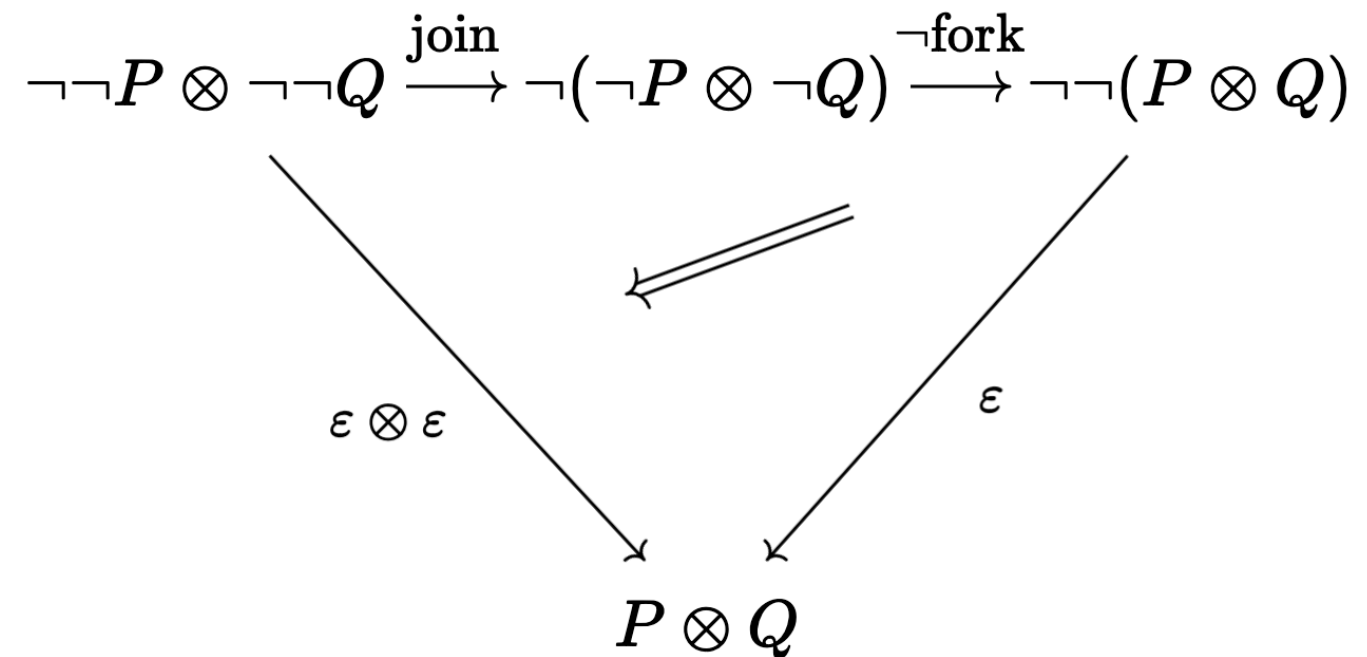
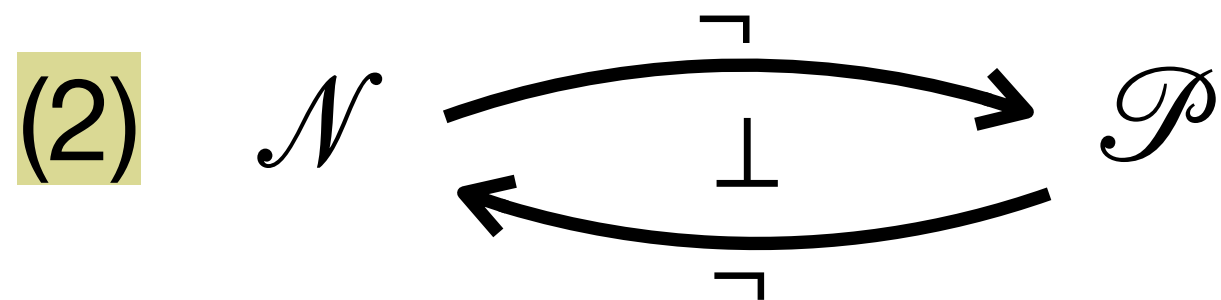
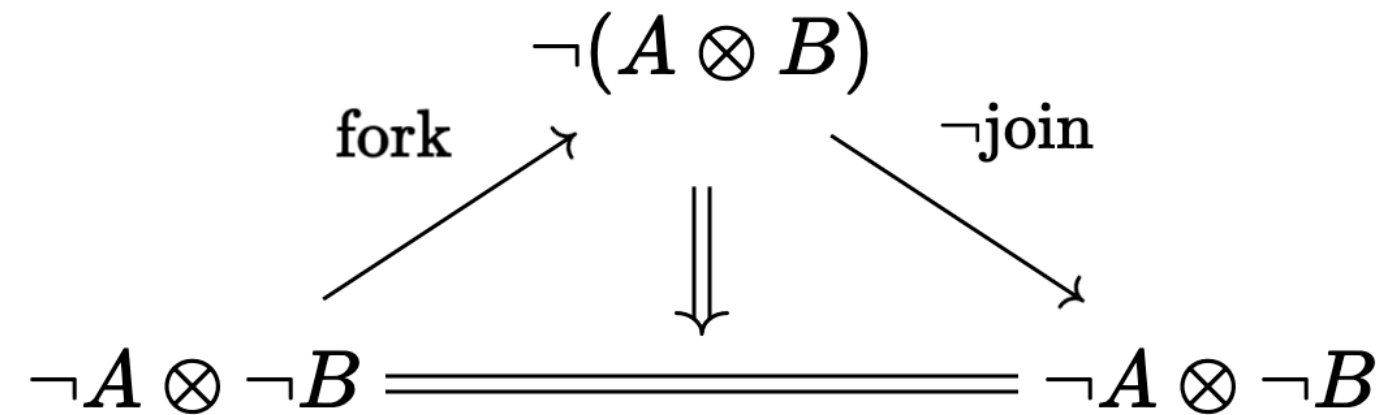
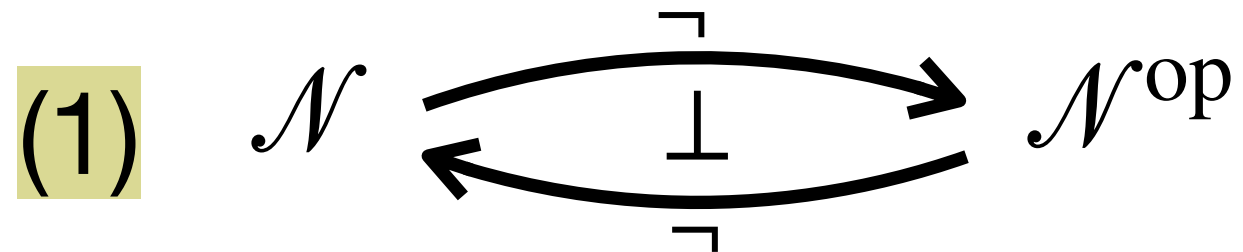
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Summary so far



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$$\begin{array}{ccc} & \neg(A \otimes B) & \\ \text{fork} \nearrow & & \searrow \neg\text{join} \\ \neg A \otimes \neg B & \Downarrow & \neg A \otimes \neg B \end{array}$$

(2) $\mathcal{N} \begin{array}{c} \xrightarrow{\neg} \\ \perp \\ \xleftarrow{\neg} \end{array} \mathcal{P}$

$$\begin{array}{ccc} \neg\neg P \otimes \neg\neg Q & \xrightarrow{\text{join}} & \neg(\neg P \otimes \neg Q) \xrightarrow{\neg\text{fork}} \neg\neg(P \otimes Q) \\ & \searrow \varepsilon \otimes \varepsilon & \swarrow \varepsilon \\ & P \otimes Q & \end{array}$$

Abstractly: The adjunction counit ε is weakly monoidal.

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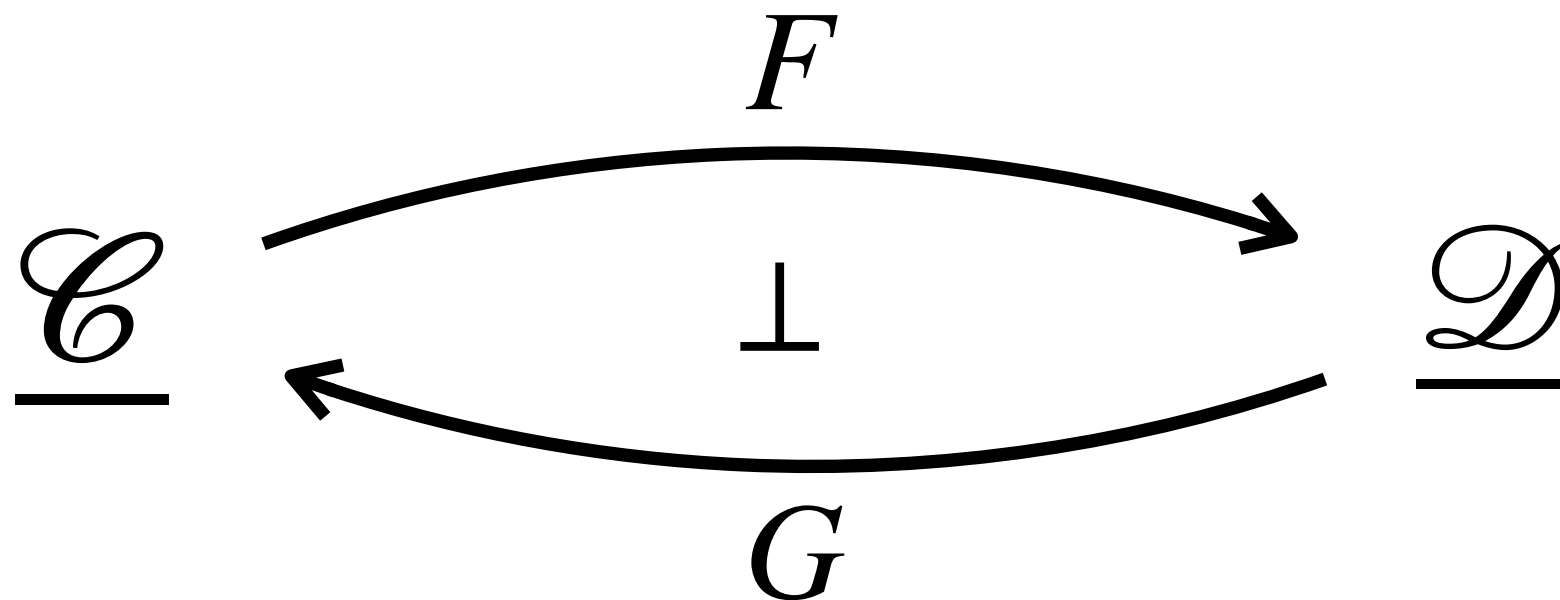
Abstractly: The adjunction counit ε is weakly monoidal.

Theorem. If a 2-adjunction has weakly monoidal counit (+ details ...) then the induced 2-monad is concurrent.

1. Examples
2. Concurrency and dialogue structure
- 3. Linear / non-linear games**

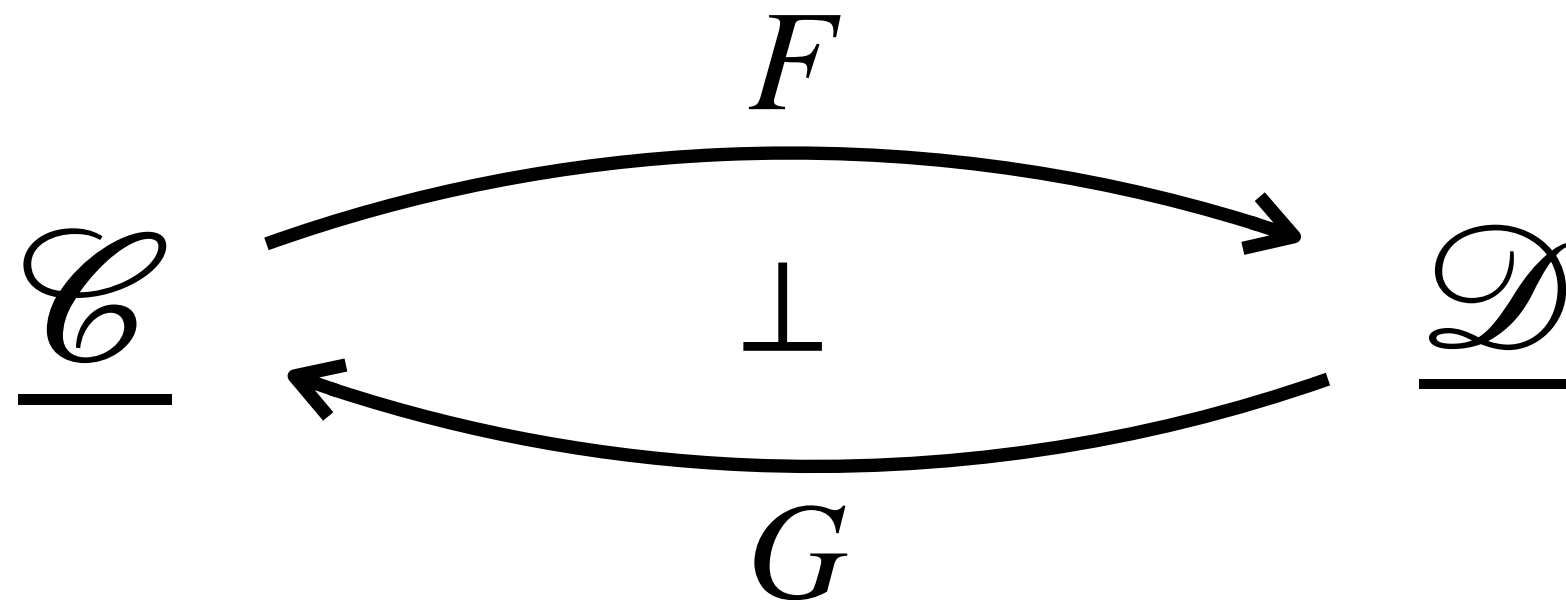
“Adjunction models”

- Includes models of: call-by-push-value, linear/non-linear logic, LLP, tensorial logic, ...
- Main feature: two classes of types.
- Induce models for call-by-value and call-by-name.



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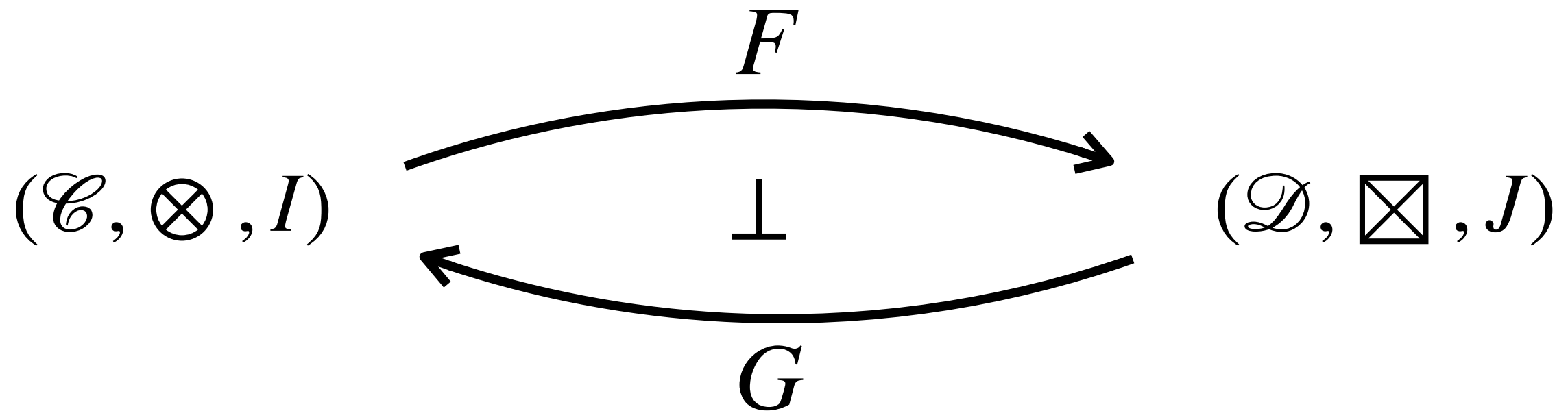


Motivation: incorporate concurrency as a 2-categorical weakening of monoidal structure (which one?).

Symmetric monoidal adjunctions

$$\begin{array}{ccc} (\mathcal{C}, \otimes, I) & \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} & (\mathcal{D}, \boxtimes, J) \end{array}$$

Symmetric monoidal adjunctions



- Monoidal structure for F and G :

$$FA \boxtimes FB \longrightarrow F(A \otimes B) \qquad GC \otimes GD \longrightarrow G(C \boxtimes D)$$

- Unit and counit are strictly monoidal
 $\implies$ concurrent structure is degenerate.
- Linear / non-linear when $\otimes$ is cartesian.

Non-linear games

Compose a dialogue 2-category with a LNL adjunction.

$$\begin{array}{ccccc}
 & L & & \neg & \\
 (\mathcal{M}, \times) & \xrightarrow{\quad} & (\mathcal{G}, \otimes) & \xrightarrow{\quad} & (\mathcal{G}, \otimes)^{\text{op}} \\
 & \perp & & \perp & \\
 & M & & \neg &
 \end{array}$$

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Prop. Adjunctions in which counit is weakly monoidal compose.

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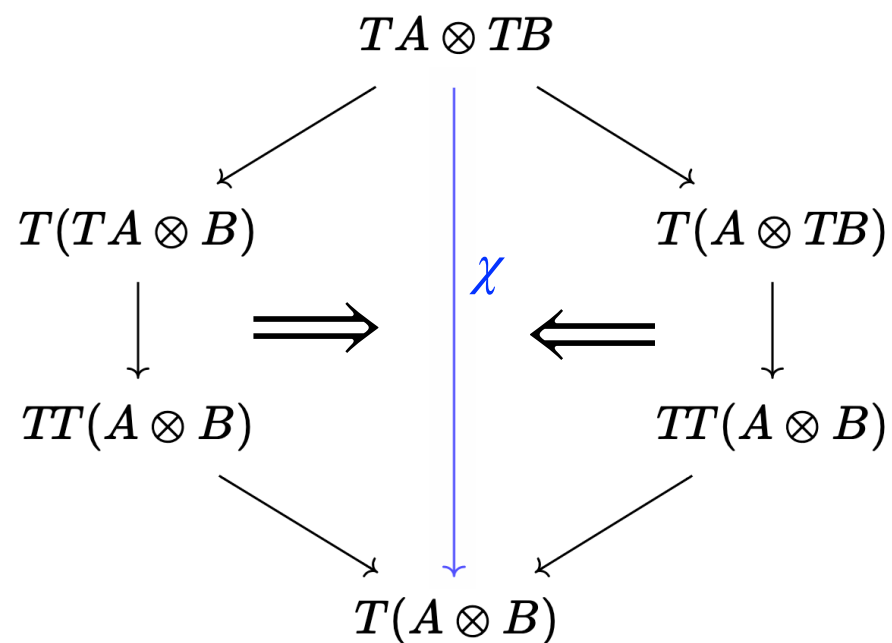
$$\begin{array}{ccccc}
 & L & & \neg & \\
 (\mathcal{M}, \times) & \xrightleftharpoons[\quad \perp \quad]{} & (\mathcal{G}, \otimes) & \xrightleftharpoons[\quad \neg \quad]{} & (\mathcal{G}, \otimes)^{\text{op}} \\
 & M & & &
 \end{array}$$

Prop. Adjunctions in which counit is weakly monoidal compose.

Corollary. Concurrent monad on non-linear category of games.

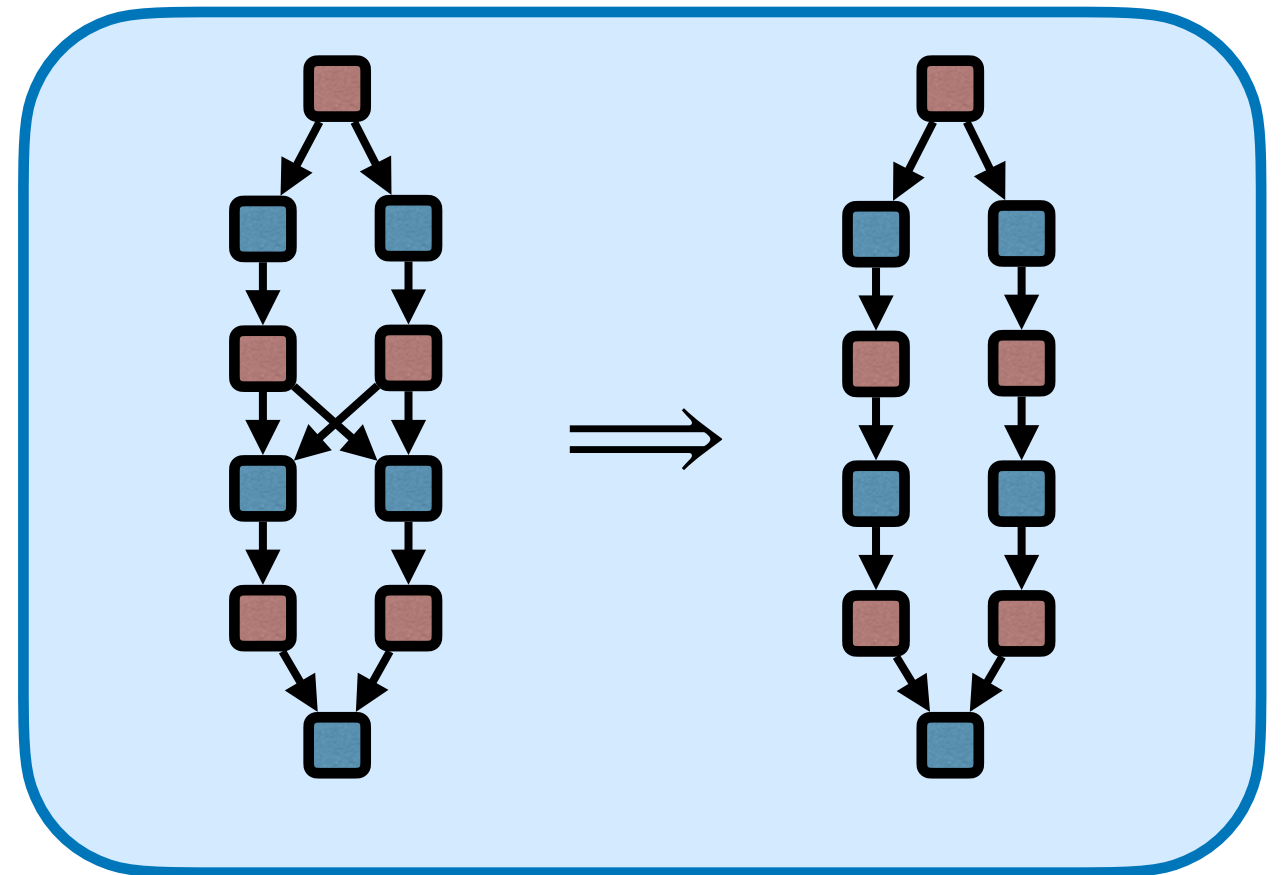
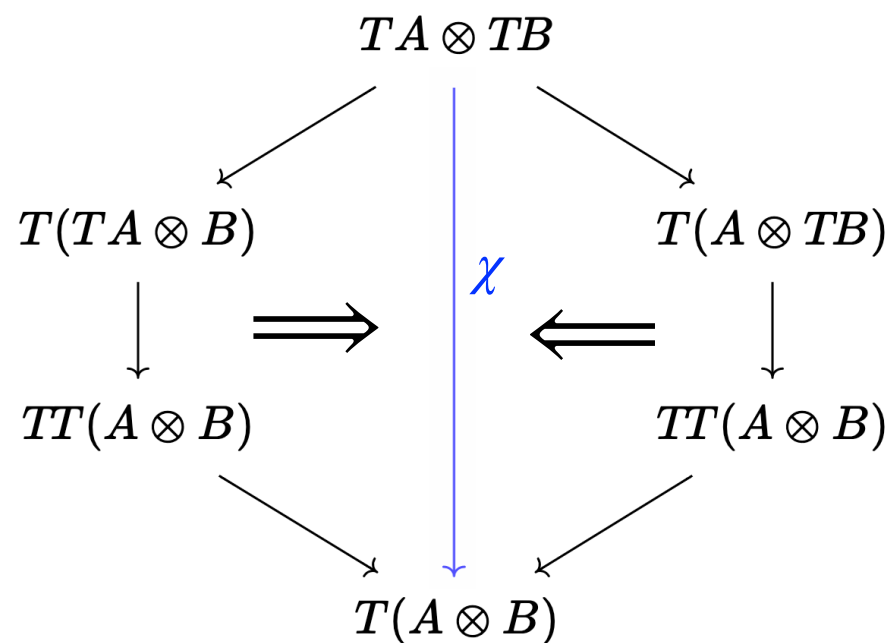
e.g. $\mathcal{G}_!$

Summary



- Model concurrency as a 2-categorical weakening of monoidal structure.
- Works for monads, linear games and non-linear games.
- TBD whether there is a general story.

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