An Adjunction Between the Lens and the Int Construction in Categorical Game Semantics

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Outline

Overview

1. Preliminary
2. The Adjunction $\tau \dashv \rho$
3. Game-Theoretic Interpretation
4. Conclusion & Future Work

Two constructions

There are two bidirectional constructions for monoidal categories:

- $\mathbf{Optic}(\mathcal{C})$ — a free teleological category on a monoidal category $\mathcal{C}$.
- $\mathbf{Int}(\mathcal{C})$ — a free compact closed category on a traced monoidal category $\mathcal{C}$.
- ($\mathbf{Lens}(\mathcal{C})$ is a special case of $\mathbf{Optic}(\mathcal{C})$ when $\mathcal{C}$ is cartesian.)

This talk

Question: How do $\mathbf{Optic}(\mathcal{C})$ and $\mathbf{Int}(\mathcal{C})$ relate?

Answer:
There always exists a canonical embedding

$$\tau : \mathbf{Optic}(\mathcal{C}) \to \mathbf{Int}(\mathcal{C})$$

, and sometimes

$$\rho : \mathbf{Int}(\mathcal{C}) \to \mathbf{Optic}(\mathcal{C})$$

such that

$$\tau \dashv \rho.$$

Traced monoidal categories

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category.

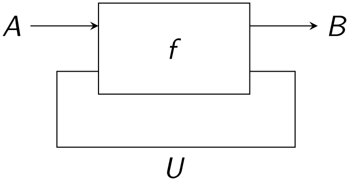
Trace

A family of maps

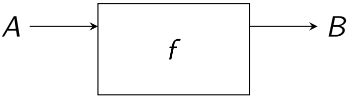
$$\mathrm{Tr}^{U}_{A,B} : \mathcal{C}(A \otimes U, B \otimes U) \longrightarrow \mathcal{C}(A, B),$$

natural in A, B, dinatural in U (with some coherence axioms).

Trace: string diagram



$$f : A \otimes U \to B \otimes U$$



$$\mathrm{Tr}^{U}_{A,B}(f) : A \to B$$

Example: $\mathbf{DCPO}$

$\mathbf{DCPO}$ (dcpo's and Scott-continuous maps) is traced cartesian.

Given $f = (f_1, f_2) : A \times U \to B \times U$, define

$$u_a := \mathrm{lfp}\big(u \mapsto f_2(a, u)\big), \qquad \mathrm{Tr}(f)(a) := f_1(a,\ u_a).$$

Trace is *least fixpoint* — the setting for possibly-nonterminating recursive computation.

$\mathbf{Optic}(\mathcal{C})$: definition

For a symmetric monoidal category $\mathcal{C}$, define the *optic category*:
Objects: (A^+, A^-), for $A^+, A^- \in \mathcal{C}$.

$$\mathbf{Optic}(\mathcal{C})\big((A^+,\ A^-),\ (B^+,\ B^-)\big) := \int^{M \in \mathcal{C}} \mathcal{C}(A^+,\ M \otimes B^+) \times \mathcal{C}(M \otimes B^-,\ A^-)$$

A morphism is a *get* $A^+ \to M \otimes B^+$ and a *put* $M \otimes B^- \to A^-$, sharing a hidden residual M.

Cartesian case: concrete lenses

When $\mathcal{C}$ is cartesian, the coend has a canonical representative $M = A^+$:

$$\mathbf{Optic}(\mathcal{C})\big((A^+, A^-), (B^+, B^-)\big) \cong \mathcal{C}(A^+, B^+) \times \mathcal{C}(A^+ \times B^-, A^-),$$

a pair (v, u) — the usual $\mathbf{Lens}$. $\mathbf{Optic}(\mathcal{C})$ is a free teleological category on $\mathcal{C}$ (Riley, 2018).

$\mathbf{Int}(\mathcal{C})$: definition

Let $\mathcal{C}$ be traced (cartesian) monoidal. Objects: (A^+, A^-), for $A^+, A^- \in \mathcal{C}$.

$$\mathbf{Int}(\mathcal{C})\big((A^+,\ A^-),\ (B^+,\ B^-)\big) := \mathcal{C}(A^+ \times B^-,\ B^+ \times A^-)$$

$\mathbf{Int}(\mathcal{C})$ is a free compact closed category on $\mathcal{C}$ (Joyal–Street–Verity, 1994).

The embedding τ: definition

There exists a canonical embedding $\tau : \mathbf{Optic}(\mathcal{C}) \to \mathbf{Int}(\mathcal{C})$ for any traced monoidal $\mathcal{C}$. On objects, τ is the identity.
On a morphism $(f, g) = \big(A^+ \to M \otimes B^+,\ M \otimes B^- \to A^-\big)$,

$$\tau(f,g) \;=\; A^+ \otimes B^- \xrightarrow{f \otimes \mathrm{id}} M \otimes B^+ \otimes B^- \xrightarrow{\text{symmetry}} B^+ \otimes M \otimes B^- \xrightarrow{\mathrm{id} \otimes g} B^+ \otimes A^-.$$

This is gotten by the universal property of $\mathbf{Optic}(\mathcal{C})$ (Riley, 2018), concretely calculated in
("OPTIC EMBEDS INTO THE INT CONSTRUCTION", Di Lavore–Román).

Overview

Main theorem

Let $\mathcal{C}$ be a traced cartesian closed category.

Theorem

There is an adjunction

$$\tau \dashv \rho$$

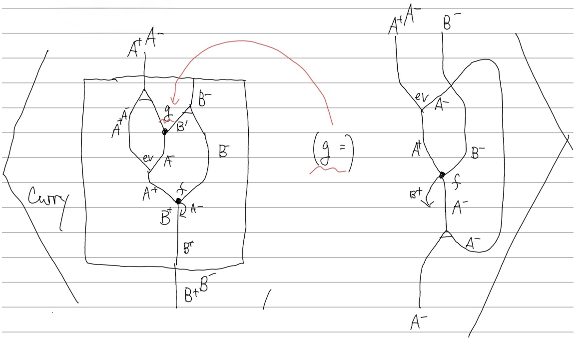
with ρ on objects given by

$$\rho(B^+, B^-) \;=\; \big([B^-, B^+],\; B^-\big).$$

, on morphisms by

$$\rho(f : A^+ \times B^- \to B^+ \times A^-) \;=\; \ldots$$

Main theorem (cont.)



τ is generic

$\tau : \mathbf{Optic}(\mathcal{C}) \to \mathbf{Int}(\mathcal{C})$ exists for *any* traced monoidal $\mathcal{C}$ (Di Lavore–Román, generalising Hedges' cartesian case).

No closed structure required — just the coend form of $\mathbf{Optic}$.

ρ is special

The above ρ needs strictly more:

- a **closed** structure, to define ρ on objects;
- the **Cartesian structure**

What is the condition for ρ to exist in general? Open question.

Known examples

- traced CCC (e.g. $\mathbf{DCPO}$)
- traced coCartesian coclosed category
- compact closed category
- Lawvere quantale $[0, \infty)$
- products of the above

Result: compact closed

If $\mathcal{C}$ is already compact closed, trace comes for free from duals, and

$$\mathbf{Int}(\mathcal{C}) \simeq \mathcal{C}.$$

Up to this equivalence, ρ **does nothing** — there is no extra fixpoint content left to construct.

Result: the Lawvere quantale $[0, \infty)$

$\mathcal{C} = \big([0, \infty), \geq, +, 0\big)$ is traced closed but **not** cartesian. The cartesian formula fails here; the actual right adjoint is

$$\rho(B^+, B^-) \;=\; \big([B^-, B^+],\; [B^+, B^-]\big).$$

Non-examples

- $\mathbf{DCPO}$ with strict maps
- $\mathbf{Rel}$ with the usual biproduct

Overview

The framework, briefly

Open games over $\mathbf{Int}(\mathbf{DCPO})$ (Hedges §11): strategy profiles, and an equilibrium condition for each context.

For a decision with strategy σ and context (h, k):

$$\mu y.\, \sigma(h(y)) \;\in\; \arg\max(k).$$

A least fixpoint — the *same shape* as ρ's morphism action in §2.

ρ as cofree causal approximation

Reading: $\mathbf{Optic}(\mathcal{C})$ is the best *causal* (feedforward) approximation of $\mathbf{Int}(\mathcal{C})$'s backward-reading strategies.

ρ is the **cofree** causal-object construction — right adjoint to forgetting causality.

Not a metaphor

Recall from §2:

$$u(h, c^-) = b^-_{h,c^-} = \mathrm{lfp}\big(b^- \mapsto q(h(b^-), c^-)\big)$$

This *is* $\mu y.\sigma(h(y)) \in \arg\max(k)$ — same fixpoint, same computation.

Matching pennies, in brief

- Over $\mathbf{Optic}$: constant strategies only — **zero** pure equilibria, as expected.
- Over $\mathbf{Int}$: second-move advantage gives **four** equilibria.
- Both players second-guessing: **deadlock** at $\perp$ — exposed as *not* an equilibrium.

(Full computation: backup slides.)

Applying ρ: program equilibrium

ρ(matching pennies) is a *reaction-function commitment game*: players commit to functions $B^- \to B^+$, not values. Resolved exactly by the fixpoint construction of §2.

Overview

Summary: the adjunction

Over $\mathcal{C} = \mathbf{DCPO}$:

$$\tau \dashv \rho$$

On objects, ρ is **currying**. On morphisms, ρ needs **trace** (least fixpoint).

Summary: where ρ exists

Confirmed over: traced CCC, traced coCCC, compact closed, the Lawvere quantale $[0, \infty)$, and products of these.

Summary: the game reading

$\mathbf{Optic}(\mathcal{C})$ = best causal approximation of $\mathbf{Int}(\mathcal{C})$'s backward-reading strategies.

Thank you

Questions?

Backup: matching pennies — the lazy case

Constant strategies $\sigma \equiv a$, $\tau \equiv b$ (including at $\perp$):

$$\mu x.\sigma(\tau(x)) = a, \qquad \mu y.\tau(\sigma(y)) = b$$

Player 1 deviates: $a \notin \arg\max_x U_1(x, b) = \{b\}$. No constant-strategy equilibrium exists — matches the classical fact that matching pennies has no pure Nash equilibrium.

Backup: the two verified equilibria

$\sigma(y) = y$, $\tau \equiv a$:

$$\mu x.\sigma(\tau(x)) = \mu y.\tau(\sigma(y)) = a$$

$a \in \arg\max_x U_1(x, a) = \{a\}$; $\{a, b\} = \arg\max_y U_2(y, y)$, and $a \in \{a, b\}$. ✓

$\sigma(y) = y$, $\tau \equiv b$: symmetric computation gives $b \in \{b\} = \arg\max_x U_1(x, b)$ and $b \in \{a, b\}$. ✓

Backup: the symmetric pair, and deadlock

By symmetry (swap the roles of the two players), two further equilibria exist with $\tau(x) = x$ and σ constant. **Deadlock:**

$\sigma(y) = y$, $\tau(x) = x$ gives

$$\mu x.\sigma(\tau(x)) = \mu y.\tau(\sigma(y)) = \perp,$$

but $\perp \notin \{a, b\} = \arg\max(\cdots)$ — **not** an equilibrium.

Backup: computing the Optic hom-set

$$\mathbf{Optic}(\mathcal{C})\big((A^+, A^-), (B^+, B^-)\big) \neq \emptyset \iff \exists M \geq 0 :\ A^+ \geq M{+}B^+ \wedge M{+}$$

Solving for M: need $A^- - B^- \leq M \leq A^+ - B^+$, $M \geq 0$. Such M exists iff $A^+ - B^+ \geq A^- - B^-$ (i.e. $A^+ - A^- \geq B^+ - B^-$) *and* $A^+ \geq B^+$.